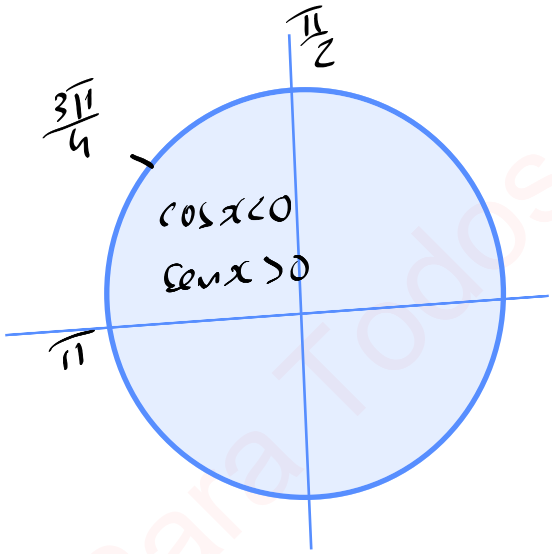
Funções Trigonométricas 4
Resolução

1)

$x \in \left] \frac{\pi}{2}, \pi \right[$



(A) $h'(x) = \cos x - \operatorname{sen} x$

(B) $h'(x) = \cos x + \operatorname{sen} x$

(C) $h'(x) = -\operatorname{sen} x - \cos x$

(D) $h'(x) = -\underset{>0}{\cos x} + \underset{>0}{\operatorname{sen} x}$

D

2.1

$$\lim_{x \to \frac{\pi}{2}} \frac{x - \frac{\pi}{2}}{\operatorname{sen}\left(\frac{2x - \pi}{2}\right)} \quad \frac{0}{0} \quad = \lim_{x \to \frac{\pi}{2}} \frac{x - \frac{\pi}{2}}{\operatorname{sen}\left(x - \frac{\pi}{2}\right)}$$

$x \to \frac{\pi}{2}$, $x - \frac{\pi}{2} \to 0$, $y = x - \frac{\pi}{2}$, $y \to 0$

$$= \lim_{y \to 0} \frac{y}{\operatorname{sen} y} = \frac{1}{\lim\limits_{y \to 0} \frac{\operatorname{sen} y}{y}} = \frac{1}{1} = 1$$

2.2 $\displaystyle \lim_{x \to 0} \frac{\tan(2x)}{\frac{x}{3}} = \frac{0}{0}$

$$= \lim_{x \to 0} \frac{3\,\frac{\operatorname{sen} 2x}{\cos 2x}}{x} =$$

$$= 3 \lim_{x \to 0} \frac{\operatorname{sen} 2x}{x \cos 2x} =$$

$$= 3 \lim_{x \to 0} \frac{1}{\cos 2x} \cdot \lim_{x \to 0} \frac{\operatorname{sen} 2x}{x}$$

$x \to 0$
$2x \to 0$

$$= 3 \cdot 2 \lim_{2x \to 0} \frac{\operatorname{sen} 2x}{2x} = 3 \cdot 2 \cdot 1 = 6$$

2.3 $\displaystyle \lim_{x \to 0} \frac{\operatorname{sen}(3x)\cos(2x) - \operatorname{sen}(2x)\cos(3x)}{\frac{x}{4}} \quad \frac{0}{0}$

$$\lim_{x \to 0} \frac{4 \operatorname{sen}(3x - 2x)}{x} = 4 \lim_{x \to 0} \frac{\operatorname{sen} x}{x} =$$

$$= 4 \cdot 1 = 4$$

3.

$$f(x)=\begin{cases}\dfrac{x-\operatorname{sen}4x}{3x} & \text{se } x<0\\[2mm] \dfrac{5x-3}{x+1}-\dfrac{1}{2} & \text{se } x\geq 0\end{cases}$$

3.1

$$\lim_{x\to 0^-}\frac{x-\operatorname{sen}4x}{3x}\overset{\frac{0}{0}}{=}\frac{1}{3}\lim_{x\to 0^-}\frac{x}{x}-\lim_{x\to 0^-}\frac{\operatorname{sen}4x}{3x}$$

$$=\frac{1}{3}-\frac{4}{3}\lim_{x\to 0^-}\frac{\operatorname{sen}4x}{4x}=\frac{1}{3}-\frac{4}{3}\lim_{4x\to 0^-}\frac{\operatorname{sen}4x}{4x}=$$

$x\to 0^-$

$4x\to 0^-$

$$=\frac{1}{3}-\frac{4}{3}\times 1=\frac{1}{3}-\frac{4}{3}=-\frac{3}{3}=-1$$

$$\lim_{x\to 0^+}\frac{5x-3}{x+1}-\frac{1}{2}=\frac{0-3}{0+1}-\frac{1}{2}=-3-\frac{1}{2}$$

$$=-\frac{7}{2}$$

Como $\lim_{x\to 0^-}f(x)\neq\lim_{x\to 0^+}f(x)$ a função não é contínua em $x=0$

3.2 f é contínua em $]-\infty, 0[$, em particular, é contínua em $\left[-\frac{\pi}{4}, -\frac{\pi}{6}\right]$

$$f\left(-\frac{\pi}{4}\right) = \frac{-\frac{\pi}{4} - \operatorname{sen}\pi}{3\cdot\left(-\frac{\pi}{4}\right)} = \frac{-\frac{\pi}{\cancel{4}}}{-\frac{3\pi}{\cancel{4}}} = \frac{1}{3} > 0$$

$$f\left(-\frac{\pi}{6}\right) = \frac{-\frac{\pi}{6} - \operatorname{sen}\left(-\frac{2\pi}{3}\right)}{-\frac{\pi}{2}} < 0$$

logo $f\left(-\frac{\pi}{4}\right) \times f\left(-\frac{\pi}{6}\right) < 0$, e f é contínua em $\left[-\frac{\pi}{4}, -\frac{\pi}{6}\right]$, pelo corolário do teorema de Bolzano-Cauchy

$$\exists x \in \left]-\frac{\pi}{4}, -\frac{\pi}{6}\right[: f(x) = 0$$

4.1 $f(x) = x - \cos 2x$, em $[0, \pi]$

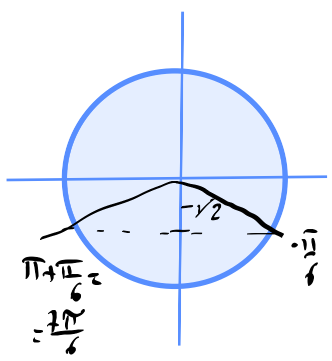
$$f'(x) = 1 + 2\operatorname{sen}(2x)$$

$$f'(x) = 0 \Leftrightarrow 1 + 2\operatorname{sen}(2x) = 0 \Leftrightarrow \operatorname{sen} 2x = -\frac{1}{2} \Leftrightarrow$$

$$\Leftrightarrow \operatorname{sen} 2x = \operatorname{sen}\left(-\frac{\pi}{6}\right) \Leftrightarrow$$

$$\Leftrightarrow 2x = -\frac{\pi}{6} + 2k\pi \lor 2x = \pi + \frac{\pi}{6} + 2k\pi,\ k \in \mathbb{Z}$$

$$\Leftrightarrow x = -\frac{\pi}{12} + k\pi \lor x = \frac{7\pi}{12} + k\pi,\ k \in \mathbb{Z}$$



$x \in [0, \pi]$

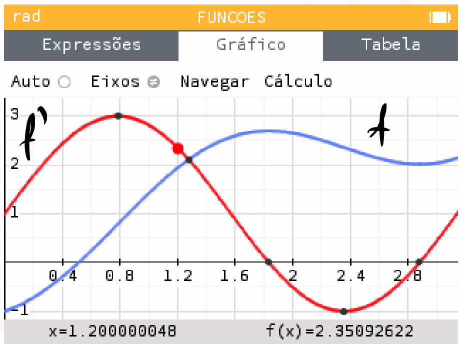
$k = 0$

$x = -\frac{\pi}{12}$ ✗ $\vee$ $x = \frac{7\pi}{12}$ ✓

$k = 1$

$x = -\frac{\pi}{12} + \pi$ $\vee$ $x = \frac{7\pi}{12} + \pi$ ✗

$x = \frac{11\pi}{12}$ ✓



	0		$\frac{7\pi}{12}$		$\frac{11\pi}{12}$		π
$f'(x)$	$+$	$+$	0	$-$	0	$+$	$+$
$f(x)$	mín	↗	Máx	↘	mín	↗	máx

f é crescente em $\left]0, \frac{7\pi}{12}\right[\cup \left]\frac{11\pi}{12}, \pi\right[$

f é decrescente em $\left]\frac{7\pi}{12}, \frac{11\pi}{12}\right[$

$f(0) = 0 - 1 = -1$ e $f\left(\frac{11\pi}{12}\right) = \frac{11\pi}{12} - \cos\left(\frac{11\pi}{6}\right) = \frac{11\pi}{12} + \frac{\sqrt{3}}{2}$
são mínimos de f

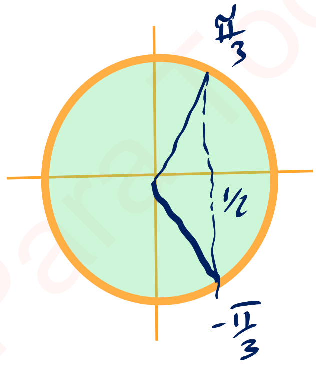
$f\left(\frac{7\pi}{12}\right) = \frac{7\pi}{12} - \cos\left(\frac{7\pi}{6}\right) = \frac{7\pi}{12} + \frac{\sqrt{3}}{2}$ e $f(\pi) = \pi + 1$
são máximos relativos de f

4.2 $f(x) = x - \operatorname{sen}(2x)$ em $[0, \pi]$

$$f'(x) = 1 - 2\cos(2x)$$

$$f'(x) = 0 \Leftrightarrow 1 - 2\cos(2x) = 0 \Leftrightarrow \cos(2x) = \frac{1}{2}$$

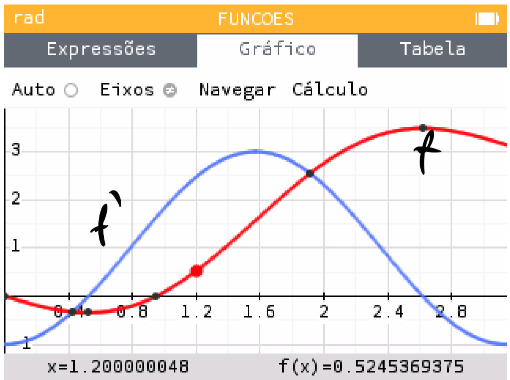
$$\cos 2x = \cos \frac{\pi}{3} \Leftrightarrow$$



$$\Leftrightarrow 2x = \frac{\pi}{3} + 2k\pi \vee 2x = -\frac{\pi}{3} + 2k\pi, k \in \mathbb{Z}$$

$$\Leftrightarrow x = \frac{\pi}{6} + k\pi \vee x = -\frac{\pi}{6} + k\pi, k \in \mathbb{Z}$$

$x \in [0, \pi]$



$k = 0$

$x = \frac{\pi}{6}$ ✓ $\vee$ $x = -\frac{\pi}{6}$ ✗

$k = 1$

$x = \frac{\pi}{6} + \pi$ ✗ $\vee$ $x = -\frac{\pi}{6} + \pi = \frac{5\pi}{6}$ ✓

	0		$\frac{\pi}{6}$		$\frac{5\pi}{6}$		π
$f'(x)$	$-$	$-$	0	$+$	0	$-$	$-$
$f(x)$	Máx	↘	min	↗	Máx	↘	min

f é crescente em $\left]\frac{\pi}{6}, \frac{5\pi}{6}\right[$

f é decrescente em $\left]0, \frac{\pi}{6}\right[$ e $\left]\frac{5\pi}{6}, \pi\right[$

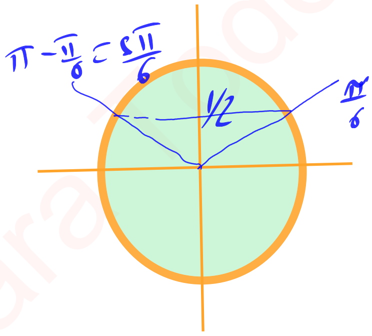
$f\left(\frac{\pi}{6}\right) = \frac{\pi}{6} + \operatorname{sen}(\pi/3) = \frac{\pi}{6} + \frac{\sqrt{3}}{2}$ e $f(\pi) = \pi$ são mínimos relativos de f

$f(0) = 0$ e $f\left(\frac{5\pi}{6}\right) = \frac{5\pi}{6} + \operatorname{sen}\left(\frac{5\pi}{3}\right)$ são máximos

5. $f'(x) = x + 2\cos x$ em $[-\pi, \pi]$

$f''(x) = 1 - 2\,\text{sen}\,x$

$f''(x) = 0 \Leftrightarrow 1 - 2\,\text{sen}\,x = 0 \Leftrightarrow \text{sen}\,x = \frac{1}{2}$



$\text{sen}\,x = \text{sen}\,\frac{\pi}{6}$

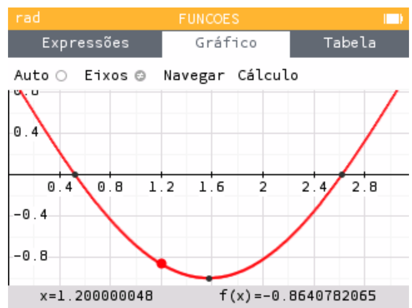
$x = \frac{\pi}{6} + 2k\pi \vee x = \frac{5\pi}{6} + 2k\pi, \quad k \in \mathbb{Z}$

$k = 0$

$x = \frac{\pi}{6}$ ✓ $\vee$ $x = \frac{5\pi}{6}$ ✓

$k = -1$

$x = \frac{\pi}{6} - 2\pi$ ✗ $\vee$ $x = \frac{5\pi}{6} - 2\pi$ ✗



x	$-\pi$		$\frac{\pi}{6}$		$\frac{5\pi}{6}$		π
f''	+	+	0	−	0	+	+
f	∪	∪	PI	∩	PI	∪	∪

f tem a concavidade voltada para cima em $[-\pi, \frac{\pi}{6}[$ e $]\frac{5\pi}{6}, \pi]$

f tem a concavidade voltada para baixo em $]\frac{\pi}{6}, \frac{5\pi}{6}[$

$f(\frac{\pi}{6})$ e $f(\frac{5\pi}{6})$ são pontos de inflexão

6.

$$f(x)=\begin{cases}\dfrac{\cos(2x-2\pi)}{x-\pi} & \text{se } x<\pi\\[2mm] \dfrac{1}{2\pi} & \text{se } x=\pi\\[2mm] \dfrac{\operatorname{sen}(\pi-x)}{\pi^2-x^2} & \text{se } x>\pi\end{cases}$$

$$\lim_{x\to\pi^-}\frac{\cos(2x-2\pi)}{x-\pi}=\frac{\cos(2\pi-2\pi)}{\pi-\pi}=\frac{\cos 0}{0^-}=\frac{1}{0^-}=-\infty$$

$$\lim_{x\to\pi^+}\frac{\operatorname{sen}(\pi-x)}{\pi^2-x^2}\overset{\left(\frac{0}{0}\right)}{=}\lim_{x\to\pi^+}\frac{\operatorname{sen}(\pi-x)}{(\pi-x)(\pi+x)}=$$

$$=\lim_{x\to\pi^+}\frac{\operatorname{sen}(\pi-x)}{\pi-x}\cdot\lim_{x\to\pi^+}\frac{1}{\pi+x}=$$

$x\to\pi^+,\ x-\pi\to0^+$

Seja $y=x-\pi \Leftrightarrow -y=\pi-x$, $y\to0$

C.A.
$\operatorname{sen}(-y)=$
$=-\operatorname{sen}y$

$$=\lim_{y\to0}\frac{\operatorname{sen}(-y)}{-y}\cdot\frac{1}{2\pi}=$$

$$=\lim_{y\to0}\frac{\operatorname{sen}y}{y}\cdot\frac{1}{2\pi}=1\times\frac{1}{2\pi}=\frac{1}{2\pi}$$

$\lim_{x\to\pi^-}f(x)\neq\lim_{x\to\pi^+}f(x)$ f não é contínua em $\mathbb{R}$.

<u>NOTA:</u> Não era necessário calcular o $\lim_{x\to\pi^+}f(x)$

Porque $\lim_{x\to\pi^-}f(x)\neq f(\pi)$

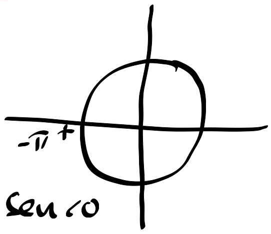
7.

$$f(x)=\begin{cases} \dfrac{2x}{\operatorname{sen} x} & \text{se } -\pi < x < 0 \\ 2 & \text{se } x = 0 \\ 1-\dfrac{2}{x^2} & \text{se } x > 0 \end{cases}$$

7.1

A.V.

$$\lim_{x\to -\pi^+} \frac{2x}{\operatorname{sen}(x)} = \frac{-2\pi}{0^-} = +\infty$$



$x=-\pi$ é uma assíntota vertical ao gráfico de f quando $x\to -\pi^+$

$$\lim_{x\to 0^-} \frac{2x}{\operatorname{sen} x} \overset{\frac{0}{0}}{=} \frac{1}{\frac{1}{2}\lim\limits_{x\to 0^-} \frac{\operatorname{sen} x}{x}} = \frac{1}{\frac{1}{2}\times 1} = 2$$

$$\lim_{x\to 0^+} 1-\frac{2}{x^2} = 1-\frac{2}{0^+} = 1-\infty = -\infty$$

$x=0$ é uma assíntota vertical ao gráfico de f quando $x\to 0^+$

A.H.

$$\lim_{x\to +\infty} 1-\frac{2}{x^2} = 1-\frac{2}{+\infty} = 1-0 = 1$$

$y=1$ é uma assíntota horizontal ao gráfico de f quando $x\to +\infty$

7.2 $f(x) - \sqrt{7} = 0 \Leftrightarrow f(x) = \sqrt{7}$

C.A.
$\sqrt{7} \approx 2,6$

f é contínua em $]-\infty, 0[$ por definição da função f, em particular é contínua em $[-2,-1]$

$$f(-2) = \frac{-4}{\operatorname{sen}(-2)} \approx 4,4 > \sqrt{7}$$

$$f(-1) = \frac{-2}{\operatorname{sen}(-1)} \approx 2,4 < \sqrt{7}$$

Assim $f(-1) < \sqrt{7} < f(-2)$ e f é contínua em $[-2,-1]$, logo pelo Teorema de Bolzano-Cauchy $\exists x \in]-2,-1[: f(x) = \sqrt{7}$

8.

$$f(x) = \begin{cases} \dfrac{k \operatorname{sen}(2x)}{x^2 - 2x} & \text{se } x < 0 \\ 2e^{-2} & \text{se } x = 0 \\ \dfrac{x^2 + 2x}{e^{x+2} - e^2} & \text{se } x > 0 \end{cases}, k \in \mathbb{R}$$

8.1 $\lim\limits_{x \to 0^-} f(x) = f(0) = 2e^{-2}$

$$\lim_{x\to 0^-} \frac{k\operatorname{sen}(2x)}{x^2-2x} \overset{\frac{0}{0}}{=} \lim_{x\to 0^-} \frac{k\operatorname{sen} 2x}{x(x-2)} =$$

$$= \lim_{x\to 0^-} \frac{k}{x-2} \times 2\lim_{x\to 0^-} \frac{\operatorname{sen} 2x}{2x}$$

$$x \to 0^-$$
$$2x \to 0^-$$

$$= \frac{k}{-2} \times 2\lim_{2x\to 0^-} \frac{\operatorname{sen} 2x}{2x} = -\frac{k}{2} \times 2 \cdot 1 =$$

$$= -k$$

Assim $-k = 2e^{-2} \Leftrightarrow k = -2e^{-2} < 0$

$$\lim_{x\to 0^+} \frac{x^2+2x}{e^{x+2}-e^2} = \lim_{x\to 0^+} \frac{x(x+2)}{e^2(e^x-1)} =$$

$$= \lim_{x\to 0^+} \frac{x+2}{e^2} \times \lim_{x\to 0^+} \frac{x}{e^x-1}$$

$$= \frac{0+2}{e^2} \times \frac{1}{\lim\limits_{x\to 0^+} \frac{e^x-1}{x}} = 2e^{-2} \times \frac{1}{1} =$$

$$= 2e^{-2}$$

logo f é contínua em $x=0$ se

$k = -2e^{-2}$

8.2 $\lim\limits_{x\to+\infty} \dfrac{x^2+2x}{e^{x+2}-e^2} \overset{\frac{\infty}{\infty}}{=} \lim\limits_{x\to+\infty} \dfrac{x^2\left(1+\frac{2}{x}\right)}{e^2\left(e^x-1\right)} =$

$$= \lim_{x\to+\infty} \frac{1+\frac{2}{x}}{\frac{e^2\left(e^x-1\right)}{x^2}} = \frac{\lim\limits_{x\to+\infty} 1+\frac{2}{x}}{e^2 \lim\limits_{x\to+\infty} \frac{e^x-1}{x^2}} =$$

$$= \frac{1+\frac{2}{+\infty}}{e^2\left[\lim\limits_{x\to+\infty} \frac{e^x}{x^2} - \lim\limits_{x\to+\infty} \frac{1}{x^2}\right]} = \frac{1}{e^2(+\infty-0)} =$$

$$= 0$$

$y=0$ é uma assíntota horizontal ao gráfico de f.

10. $f(x) = \dfrac{\operatorname{sen}(4x)\operatorname{sen}(2x) - \operatorname{sen}^2(2x)}{\cos(4x)-1}$ $\quad x \in \left]\frac{\pi}{4}, \pi\right[\setminus \left\{\frac{\pi}{2}\right\}$

10.1 $f(x) = \frac{1}{2} - \cos(2x)$

$$\frac{\operatorname{sen}(2\cdot 2x)\operatorname{sen}(2x) - \operatorname{sen}^2(2x)}{\cos(2\cdot 2x) - 1} =$$

$$= \frac{2\operatorname{sen}(2x)\cos(2x)\operatorname{sen}(2x) - \operatorname{sen}^2(2x)}{\cos^2(2x) - \operatorname{sen}^2(2x) - 1} =$$

$$= \frac{2\operatorname{sen}^2(2x)\cos(2x) - \operatorname{sen}^2(2x)}{-\operatorname{sen}^2(2x) - \operatorname{sen}^2(2x)}$$

$-\operatorname{sen}^2(2x) =$
$= \cos^2(2x) - 1$

$$= \frac{\cancel{\operatorname{sen}^2(2x)}\,(2\cos(2x)-1)}{-2\,\cancel{\operatorname{sen}^2(2x)}} = \frac{1}{2} - \cos(2x) \quad \text{c.q.m.}$$

10.2

$$f'(x) = 2\operatorname{sen}(2x)$$

$$f'\left(\frac{\pi}{3}\right) = 2\operatorname{sen}\left(\frac{2\pi}{3}\right) = \frac{\cancel{2}\sqrt{3}}{\cancel{2}} = \sqrt{3}$$

$$y = mx + b \Leftrightarrow y = \sqrt{3}\,x + b$$

$$f\left(\frac{\pi}{3}\right) = \frac{1}{2} - \cos\left(\frac{2\pi}{3}\right) = \frac{1}{2} + \frac{1}{2} = 1$$

$$1 = \sqrt{3}\cdot\left(\frac{\pi}{3}\right) + b \Leftrightarrow b = 1 - \frac{\sqrt{3}}{3}\pi$$

$$y = \sqrt{3}\,x + 1 - \frac{\sqrt{3}}{3}\pi$$

10.3

$$g(x) = f(x) + x = \frac{1}{2} - \cos(2x) + x$$

$$g'(x) = 1 + 2\operatorname{sen}(2x)$$

$$g'(x) = 0 \Leftrightarrow 1 + 2\operatorname{sen}(2x) = 0 \Leftrightarrow \operatorname{sen}(2x) = -\frac{1}{2}$$

$$\Leftrightarrow \operatorname{sen}(2x) = \operatorname{sen}\left(-\frac{\pi}{6}\right) \Leftrightarrow$$

$$\Leftrightarrow 2x = -\frac{\pi}{6} + 2k\pi \ \vee\ 2x = \pi + \frac{\pi}{6} + 2k\pi, \quad k \in \mathbb{Z}$$

$$\Leftrightarrow x = -\frac{\pi}{12} + k\pi \ \vee\ x = \frac{7\pi}{12} + k\pi, \quad k \in \mathbb{Z}$$

$$x \in \left]\frac{\pi}{2}, \pi\right[$$

$k = 0$

$$x = -\frac{\pi}{12}\ ✗ \quad \vee \quad x = \frac{7\pi}{12}\ ✓$$

$k = 1$

$x = -\frac{\pi}{12} + \pi \quad \cup \quad x = \frac{7\pi}{12} + \pi$ ✗

$x = \frac{11\pi}{12}$ ✓

x	$\frac{\pi}{2}$		$\frac{7\pi}{12}$		$\frac{11\pi}{12}$		π
$f'(x)$	$+$	$+$	0	$-$	0	$+$	$+$
$f(x)$	Mín	$\nearrow$	máx	$\searrow$	mín	$\nearrow$	Máx

f é crescente em $\left[\frac{\pi}{2}, \frac{7\pi}{12}\right[$ e $\left]\frac{11}{12}\pi, \pi\right]$

f é decrescente em $\left]\frac{7\pi}{12}, \frac{11\pi}{12}\right[$

f tem máximos relativos em $x = \frac{7\pi}{12}$ e $x = \pi$

f tem mínimos relativos em $x = \frac{\pi}{2}$ e $x = \frac{11\pi}{12}$