



$$|z_2| = |2 - 2i| = \sqrt{2^2 + (-2)^2} = \sqrt{8} = 2\sqrt{2}$$

Sendo  $\text{Arg}(2 - 2i) = \theta$ , tem-se  $\tan \theta = \frac{-2}{2} \wedge \theta \in 4.^\circ \text{Q}$ , ou

seja,  $\tan \theta = -1 \wedge \theta \in 4.^\circ$  quadrante, então  $\theta = -\frac{\pi}{4}$ .

Portanto,  $z_2 = 2\sqrt{2} e^{-\frac{\pi i}{4}}$ .

$$z_1 = (2 - 2i)^3 = \left(2\sqrt{2} e^{-\frac{\pi i}{4}}\right)^3 = (2\sqrt{2})^3 e^{-\frac{3\pi i}{4}} = 16\sqrt{2} e^{-\frac{3\pi i}{4}}$$

**2.1.**  $z_1^2 + bz_1 + c = 0 \Leftrightarrow (-1 + \sqrt{3}i)^2 + b(-1 + \sqrt{3}i) + c = 0 \Leftrightarrow$

$$\Leftrightarrow (-1 + \sqrt{3}i)^2 + b(-1 + \sqrt{3}i) + c = 0 \Leftrightarrow$$

$$\Leftrightarrow 1 - 2\sqrt{3}i + 3i^2 - b + \sqrt{3}bi + c = 0$$

$$\Leftrightarrow 1 - 2\sqrt{3}i - 3 - b + \sqrt{3}bi + c = 0 \Leftrightarrow$$

$$\Leftrightarrow -2 - b + c + (\sqrt{3}b - 2\sqrt{3})i = 0$$

$$\Leftrightarrow -2 - b + c = 0 \wedge \sqrt{3}b - 2\sqrt{3} = 0 \Leftrightarrow c = 2 + b \wedge b = 2 \Leftrightarrow$$

$$\Leftrightarrow c = 4 \wedge b = 2 \Leftrightarrow$$

Portanto,  $b = 2 \wedge c = 4$

**2.2.**  $z_1 = -1 + \sqrt{3}i$

$$|z_1| = |-1 + \sqrt{3}i| = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{4} = 2$$

Se  $\text{Arg}(-1 + \sqrt{3}i) = \theta$ :

$$\begin{cases} \tan \theta = \frac{\sqrt{3}}{-1} \Rightarrow \theta = \frac{2\pi}{3}, \text{ logo } z_1 = 2 e^{i\frac{2\pi}{3}}. \\ \theta \in 2.^\circ \text{Q} \end{cases}$$

$$\frac{z_1}{z_2} = \frac{2e^{i\frac{2\pi}{3}}}{e^{-i\alpha}} = 2e^{i(\frac{2\pi}{3} + \alpha)}$$

$\frac{z_1}{z_2}$  é um número imaginário puro de coeficiente negativo se

$$\frac{2\pi}{3} + \alpha = -\frac{\pi}{2} + 2k\pi, k \in \mathbb{Z} \Leftrightarrow \alpha = -\frac{\pi}{2} - \frac{2\pi}{3} + 2k\pi, k \in \mathbb{Z} \Leftrightarrow$$

$$\Leftrightarrow \alpha = -\frac{7\pi}{6} + 2k\pi, k \in \mathbb{Z}$$

Como  $\alpha \in [-\pi, \pi]$ ,  $\alpha = -\frac{7\pi}{6} + 2\pi = \frac{5\pi}{6}$ .

**3.1.** Sejam  $z_1 = \sqrt{2}i - \sqrt{2}$  e  $z_2 = -\sqrt{3} - i$ .

$$|z_1| = |\sqrt{2}i - \sqrt{2}| = \sqrt{(-\sqrt{2})^2 + (\sqrt{2})^2} = \sqrt{4} = 2$$

Sendo  $\text{Arg}(-\sqrt{2} + \sqrt{2}i) = \theta$ :

$$\begin{cases} \tan \theta = \frac{\sqrt{2}}{-\sqrt{2}} = -1 \Rightarrow \theta = \frac{3\pi}{4} \\ \theta \in 2.^\circ \text{Q} \end{cases}$$

Portanto,  $z_1 = 2e^{i\frac{3\pi}{4}}$ .

$$|z_2| = |-\sqrt{3} - i| = \sqrt{(-\sqrt{3})^2 + (-1)^2} = \sqrt{4} = 2$$

Se  $\beta = \text{Arg}(-\sqrt{3} - i)$ :

$$\begin{cases} \tan \beta = \frac{-1}{-\sqrt{3}} = \frac{\sqrt{3}}{3} \Rightarrow \beta = -\frac{5\pi}{6} \\ \beta \in 3.^\circ \text{Q} \end{cases}$$

Portanto,  $z_2 = 2e^{-i\frac{5\pi}{6}}$ .

$$w = \frac{z_1}{z_2} = \frac{2e^{i\frac{3\pi}{4}}}{2e^{-i\frac{5\pi}{6}}} = e^{i(\frac{3\pi}{4} + \frac{5\pi}{6})} = e^{i\frac{19\pi}{12}} = e^{-i\frac{5\pi}{12}} \quad \left| \frac{19\pi}{12} - 2\pi = -\frac{5\pi}{12} \right.$$

**3.2.**  $e^{-i\frac{5\pi}{12}} = \cos\left(-\frac{5\pi}{12}\right) + i \sin\left(-\frac{5\pi}{12}\right)$ .

Por outro lado:

$$w = \frac{\sqrt{2}i - \sqrt{2}}{-\sqrt{3} - i} = \frac{(\sqrt{2}i - \sqrt{2})(-\sqrt{3} + i)}{(-\sqrt{3} - i)(-\sqrt{3} + i)} = \frac{-\sqrt{6}i - \sqrt{2} + \sqrt{6} - \sqrt{2}i}{(-\sqrt{3})^2 - i^2}$$

$$\frac{\sqrt{6} - \sqrt{2}}{4} + \frac{-\sqrt{6} - \sqrt{2}}{4}i$$

então, vem:

$$\cos\left(-\frac{5\pi}{12}\right) + i \sin\left(-\frac{5\pi}{12}\right) = \frac{\sqrt{6} - \sqrt{2}}{4} + \frac{-\sqrt{6} - \sqrt{2}}{4}i$$

$$\Leftrightarrow \cos\left(-\frac{5\pi}{12}\right) = \frac{\sqrt{6} - \sqrt{2}}{4} \wedge \sin\left(-\frac{5\pi}{12}\right) = \frac{-\sqrt{6} - \sqrt{2}}{4}$$

$$\sin\left(\frac{\pi}{12}\right) = \sin\left(\frac{6\pi}{12} - \frac{5\pi}{12}\right) = \sin\left(\frac{\pi}{2} - \frac{5\pi}{12}\right) =$$

$$= \cos\left(-\frac{5\pi}{12}\right) = \frac{\sqrt{6} - \sqrt{2}}{4}$$

$$\cos\left(\frac{\pi}{12}\right) = \cos\left(\frac{6\pi}{12} - \frac{5\pi}{12}\right) = \cos\left(\frac{\pi}{2} - \frac{5\pi}{12}\right) =$$

$$= -\sin\left(-\frac{5\pi}{12}\right) = \frac{\sqrt{6} + \sqrt{2}}{4}$$

Logo:

$$\cos\left(\frac{\pi}{12}\right) + \sin\left(\frac{\pi}{12}\right) = \frac{\sqrt{6} + \sqrt{2}}{4} + \frac{\sqrt{6} - \sqrt{2}}{4} = \frac{2\sqrt{6}}{4} = \frac{\sqrt{6}}{2}$$

**3.3.**  $w^n = \left(e^{-i\frac{5\pi}{12}}\right)^n = e^{-i\frac{5\pi}{12}n}$

O afixo de  $w^n$ , no plano complexo, pertence ao semi eixo

positivo  $Ox$  se  $-\frac{5\pi}{12}n = 2k\pi, k \in \mathbb{Z} \wedge n \in \mathbb{N}$ .

$$-\frac{5\pi}{12}n = 2k\pi, k \in \mathbb{Z} \wedge n \in \mathbb{N} \Leftrightarrow$$

$$\Leftrightarrow n = -\frac{12}{5\pi} \times 2k\pi, k \in \mathbb{Z} \wedge n \in \mathbb{N} \Leftrightarrow$$

$$\Leftrightarrow n = -\frac{24k}{5}, k \in \mathbb{Z} \wedge n \in \mathbb{N}$$

O menor valor de  $n$  natural, nestas condições, obtém-se quando  $k = -5$ , tendo-se  $n = 24$ .

**4.1.**  $\sum_{k=0}^{30} \left[ 2e^{i\left(\frac{\pi}{6} + \frac{k\pi}{3}\right)} \right] = 2 \sum_{k=0}^{30} \left[ e^{i\left(\frac{\pi}{6} + \frac{k\pi}{3}\right)} \right] = 2 \sum_{k=0}^{30} \left[ e^{i\frac{\pi}{6}} \times e^{i\frac{k\pi}{3}} \right]$

$$= 2e^{i\frac{\pi}{6}} \times \sum_{k=0}^{30} \left( e^{i\frac{\pi}{3}} \right)^k = 2e^{i\frac{\pi}{6}} \times \left( e^0 + e^{i\frac{\pi}{3}} + e^{i\frac{2\pi}{3}} + \dots + e^{i\frac{29\pi}{3}} + e^{i\frac{30\pi}{3}} \right)$$



$e^0, e^{i\frac{\pi}{3}}, e^{i\frac{2\pi}{3}}, \dots, e^{i\frac{29\pi}{3}}$  e  $e^{i\frac{30\pi}{3}}$  são os primeiros 31 termos de

uma progressão geométrica de razão  $e^{i\frac{\pi}{3}}$  e cujo primeiro termo é  $e^0$ . Assim:

$$\begin{aligned} 2e^{i\frac{\pi}{6}} \times \sum_{k=0}^{30} \left( e^{i\frac{\pi}{3}} \right)^k &= 2e^{i\frac{\pi}{6}} \times \left| e^0 \times \frac{1 - \left( e^{i\frac{\pi}{3}} \right)^{31}}{1 - e^{i\frac{\pi}{3}}} \right| \\ &= 2e^{i\frac{\pi}{6}} \times \left( 1 \times \frac{1 - e^{i\frac{31\pi}{3}}}{1 - e^{i\frac{\pi}{3}}} \right) = \left| \frac{31\pi}{3} = 10\pi + \frac{\pi}{3} \right| \\ &= 2e^{i\frac{\pi}{6}} \times \left( 1 - \frac{e^{i\frac{\pi}{3}}}{1 - e^{i\frac{\pi}{3}}} \right) = 2e^{i\frac{\pi}{6}} \times 1 = 2 \left( \cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right) = \\ &= 2 \left( \frac{\sqrt{3}}{2} + \frac{1}{2}i \right) = \sqrt{3} + i \end{aligned}$$

4.2. Tem-se:

$$\begin{aligned} z_2 &= \frac{i^{74} - 2i^{11}}{i-3} \Leftrightarrow z_2 = \frac{i^{18 \times 4 + 2} - 2i^{2 \times 4 + 3}}{i-3} \Leftrightarrow z_2 = \frac{i^2 - 2i^3}{i-3} \\ &\Leftrightarrow z_2 = \frac{-1 - 2(-i)}{i-3} \Leftrightarrow z_2 = \frac{-1 + 2i}{-3+i} \Leftrightarrow \\ &\Leftrightarrow z_2 = \frac{(-1+2i)(-3-i)}{(-3+i)(-3-i)} \Leftrightarrow z_2 = \frac{3+i-6i-2i^2}{(-3)^2 - i^2} \Leftrightarrow \\ &\Leftrightarrow z_2 = \frac{5-5i}{10} \Leftrightarrow z_2 = \frac{1}{2} - \frac{1}{2}i \end{aligned}$$

$$|z_2| = \left| \frac{1}{2} - \frac{1}{2}i \right| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{1}{4}} = \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2}$$

Se  $\text{Arg}\left(\frac{1}{2} - \frac{1}{2}i\right) = \theta$ ,

$$\begin{cases} \tan \theta = \frac{-\frac{1}{2}}{\frac{1}{2}} = -1 \Rightarrow \theta = -\frac{\pi}{4} \rightarrow \text{Assim, } z_2 = \frac{\sqrt{2}}{2} e^{-i\frac{\pi}{4}} \\ \theta \in 4.^\circ Q \end{cases}$$

4.3. Seja  $w = |w|e^{i\theta}$ . Então,  $\bar{w} = |w|e^{-i\theta}$ . Substituindo na equação:

$$\begin{aligned} w^3 \times \bar{w} &= (z_2)^4 \Leftrightarrow (|w|e^{i\theta})^3 \times (|w|e^{-i\theta}) = \left( \frac{\sqrt{2}}{2} e^{-i\frac{\pi}{4}} \right)^4 \Leftrightarrow \\ &\Leftrightarrow |w|^3 e^{3i\theta} \times |w| e^{-i\theta} = \left( \frac{\sqrt{2}}{2} \right)^4 e^{-i\pi} \Leftrightarrow |w|^4 e^{2i\theta} = \left( \frac{\sqrt{2}}{2} \right)^4 e^{-i\pi} \\ &\Leftrightarrow |w|^4 = \left( \frac{\sqrt{2}}{2} \right)^4 \wedge 2\theta = -\pi + 2k\pi, k \in \mathbb{Z} \Leftrightarrow \\ &\Leftrightarrow |w| = \frac{\sqrt{2}}{2} \wedge \theta = -\frac{\pi}{2} + k\pi, k \in \mathbb{Z} \Leftrightarrow \\ &\Leftrightarrow w = \frac{\sqrt{2}}{2} e^{-i\frac{\pi}{2}} \vee w = \frac{\sqrt{2}}{2} e^{i\frac{\pi}{2}} \Leftrightarrow \\ &\Leftrightarrow w = -\frac{\sqrt{2}}{2}i \vee w = \frac{\sqrt{2}}{2}i \end{aligned}$$

$$S = \left\{ -\frac{\sqrt{2}}{2}i, \frac{\sqrt{2}}{2}i \right\}$$

5.1.  $w = \frac{z_1}{z_2} \Leftrightarrow w = \frac{1+2ie^{i\frac{5\pi}{6}}}{1+i\sqrt{3}}$

$$\begin{aligned} z_1 &= 1 + 2ie^{i\frac{5\pi}{6}} = 1 + 2i \left( \cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \right) = \\ &= 1 + 2i \left( -\frac{\sqrt{3}}{2} + \frac{1}{2}i \right) = 1 - \sqrt{3}i - 1 = -\sqrt{3}i = \sqrt{3}e^{-i\frac{\pi}{2}} \\ |z_2| &= \sqrt{1^2 + (\sqrt{3})^2} = 2 \end{aligned}$$

Se  $\text{Arg}(1+i\sqrt{3}) = \theta$ , tem-se:

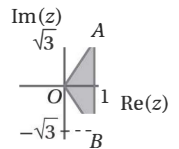
$$\begin{cases} \tan \theta = \frac{\sqrt{3}}{1} \Rightarrow \theta = \frac{\pi}{3} \text{ . Assim, } z_2 = 2e^{i\frac{\pi}{3}} \text{ .} \\ \theta \in 1.^\circ Q \end{cases}$$

$$w = \frac{\sqrt{3}e^{-i\frac{\pi}{2}}}{2e^{i\frac{\pi}{3}}} = \frac{\sqrt{3}}{2} e^{-i\frac{\pi}{2}-i\frac{\pi}{3}} = \frac{\sqrt{3}}{2} e^{-i\frac{5\pi}{6}}$$

5.2. Tem-se que  $z_2 = 1+i\sqrt{3}$ , logo  $z = \bar{z}_2 = 1-i\sqrt{3}$ .

Representando o triângulo  $[AOB]$ , no plano complexo:

$$\begin{aligned} P_{[AOB]} &= \overline{OA} + \overline{OB} + \overline{AB} = \\ &= |z_2| + |z| + 2\sqrt{3} = \\ &= 2 + 2 + 2\sqrt{3} = \\ &= 4 + 2\sqrt{3} \text{ u.c.} \end{aligned}$$



6.1. Tem-se que a multiplicação de um número complexo por

$i = e^{i\frac{\pi}{2}}$  corresponde à rotação do seu afixo segundo um ângulo de amplitude  $\frac{\pi}{2}$  radianos, no sentido positivo. Assim, e sendo  $B, C$  e  $D$  os afixos os números complexos  $z_2, z_3$  e  $z_4$ , respectivamente, tem-se:

$$\begin{aligned} z_2 &= iz_1 \Leftrightarrow z_2 = i(\sqrt{2} + 2\sqrt{2}i) \Leftrightarrow z_2 = -2\sqrt{2} + \sqrt{2}i \\ z_3 &= iz_2 \Leftrightarrow z_3 = i(-2\sqrt{2} + \sqrt{2}i) \Leftrightarrow z_3 = -\sqrt{2} - 2\sqrt{2}i \\ z_4 &= iz_3 \Leftrightarrow z_4 = i(-\sqrt{2} - 2\sqrt{2}i) \Leftrightarrow z_4 = 2\sqrt{2} - \sqrt{2}i \end{aligned}$$

Portanto,  $B(-2\sqrt{2}, \sqrt{2})$ ,  $C(-\sqrt{2}, -2\sqrt{2})$  e  $D(2\sqrt{2}, -\sqrt{2})$ .

6.2 Área pedida = Área do círculo de centro  $O$  e raio  $[OA]$  – Área do quadrado  $[ABCD]$

Área do círculo de centro  $O$  e raio  $[OA] = \pi \times (\sqrt{10})^2 = 10\pi$ .

Como  $[ABCD]$  é um quadrado de centro em  $O$ , o triângulo  $[AOD]$  é retângulo em  $O$ .

Pelo Teorema de Pitágoras:

$$\overline{AD}^2 = \overline{OA}^2 + \overline{OD}^2 = (\sqrt{10})^2 + (\sqrt{10})^2 = 20$$

Logo, a área do quadrado  $[ABCD]$  é igual a 20 u.a.e a área pedida é igual a  $(10\pi - 20)$  u.a.



$$\begin{aligned}
 (2-i)^2 + 4e^{2i} + 3\sqrt{3}i + 8e^{i\frac{\pi}{4}} &= \\
 &= \frac{3-4i+4i+3\sqrt{3}i}{3e^{i\frac{\pi}{12}}} + 8\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) = \\
 &= \frac{3+3\sqrt{3}i}{3e^{i\frac{\pi}{12}}} + 4\sqrt{2} + 4\sqrt{2}i = \frac{3(1+\sqrt{3}i)}{3e^{i\frac{\pi}{12}}} + 4\sqrt{2} + 4\sqrt{2}i = \\
 &= \frac{1+\sqrt{3}i}{e^{i\frac{\pi}{12}}} + 4\sqrt{2} + 4\sqrt{2}i = \frac{2e^{i\frac{\pi}{3}}}{e^{i\frac{\pi}{12}}} + 4\sqrt{2} - 4\sqrt{2}i \quad \begin{array}{l} z=1+\sqrt{3}i \Rightarrow \theta = \text{Arg } z \\ |z| = \sqrt{1+3} = 2 \\ \left\{ \begin{array}{l} \tan \theta = \sqrt{3} \\ \theta \in 1^\circ Q \end{array} \right. \Rightarrow \theta = \frac{\pi}{3} \end{array} \\
 &= 2e^{i\left(\frac{\pi}{3}-\frac{\pi}{12}\right)} + 4\sqrt{2} + 4\sqrt{2}i = 2e^{i\frac{\pi}{4}} + 4\sqrt{2} - 4\sqrt{2}i \\
 &= 2\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) + 4\sqrt{2} + 4\sqrt{2}i = \sqrt{2} + \sqrt{2}i + 4\sqrt{2} + 4\sqrt{2}i = \\
 &= 5\sqrt{2} + 5\sqrt{2}i
 \end{aligned}$$

$$\begin{aligned}
 \text{8. } w_1 &= i^{12n-9}(20\sqrt{3}-20i) = i^{12n} \times i^{-9} \times (20\sqrt{3}-20i) = \\
 &= (i^4)^{3n} \times i^{-9+4 \times 3} \times (20\sqrt{3}-20i) = 1 \times i^3 \times (20\sqrt{3}-20i) = \\
 &= -i \times (20\sqrt{3}-20i) = -20 - 20\sqrt{3}i
 \end{aligned}$$

Assim:

$$|w_1| = |-20 - 20\sqrt{3}i| = \sqrt{(-20)^2 + (-20\sqrt{3})^2} = \sqrt{400 + 1200} = 40$$

Se  $\theta = \text{Arg}(-20 - 2\sqrt{3}i)$ :

$$\begin{cases} \tan \theta = \frac{-20\sqrt{3}}{-20} \Rightarrow \theta = -\frac{2\pi}{3}, \text{ logo } w_1 = 40e^{-i\frac{2\pi}{3}}. \\ \theta \in 3^\circ Q \end{cases}$$

Por outro lado,  $w_2$  é uma raiz cúbica de  $w_1$  quando e apenas quando  $w_1 = (w_2)^3$ .

$$\begin{aligned}
 (w_2)^3 &= \left(2\sqrt[3]{5}e^{i\frac{10\pi}{9}}\right)^3 = \left(2\sqrt[3]{5}\right)^3 e^{i\frac{30\pi}{9}} = 8 \times 5 e^{i\frac{10\pi}{3}} = 40e^{-i\frac{2\pi}{3}} = \\
 &= w_1 \quad \left| \frac{10\pi}{3} = -\frac{2\pi}{3} + 4\pi \right.
 \end{aligned}$$

Como  $w_1 = (w_2)^3$ , podemos concluir que de facto  $w_2$  é a única raiz cúbica de  $w_1$ , como queríamos mostrar.



$$9. w = \frac{(\sqrt{3} + i)^2}{\left(2e^{i\frac{\pi}{3}}\right)^3 + 2} = \frac{-i(3 + 2\sqrt{3}i + i^2)}{2^3 e^{i\pi} + 2} = \frac{-i(3 + 2\sqrt{3}i - 1)}{-8 + 2} = \frac{-i(2 + 2\sqrt{3}i)}{-6} = \frac{2\sqrt{3} - 2i}{-6} = -\frac{\sqrt{3}}{3} + \frac{1}{3}i$$

$$|w| = \sqrt{\frac{3}{9} + \frac{1}{9}} = \sqrt{\frac{4}{9}} = \frac{2}{3}. \text{ Se } \theta = \text{Arg } w:$$

$$\begin{cases} \tan \theta = \frac{1}{-\sqrt{3}} = -\frac{\sqrt{3}}{3} \Rightarrow \theta = \frac{5\pi}{6} \rightarrow w = \frac{2}{3} e^{i\frac{5\pi}{6}} \\ \theta \in 2.^\circ Q \end{cases}$$

O afixo de  $w$  está no interior da circunferência, pois  $|w| = \frac{2}{3}$

e  $\frac{2}{3} < 2$ . A inclinação da reta  $r$  é  $\frac{5\pi}{6}$  e  $\text{Arg}(w) = \frac{5\pi}{6}$ ,

portanto, o afixo de  $w$  está sobre a reta  $r$ .

$$10.1 |z_1| = 1 \text{ e } z_2 \times \bar{z}_2 = |z_2|^2 = (3\sqrt{2})^2 = 18$$

$$\begin{aligned} \frac{\sqrt{2}|z_1|}{z_1} - \frac{z_2 \times \bar{z}_2}{\sqrt{2}i} &= \frac{\sqrt{2} \times 1}{e^{i\frac{2\pi}{3}}} - \frac{18}{\sqrt{2}i} = \sqrt{2} e^{-i\frac{2\pi}{3}} - \frac{18 \times (-\sqrt{2}i)}{\sqrt{2}i \times (-\sqrt{2}i)} \\ &= \sqrt{2} \left( \cos \frac{-2\pi}{3} + i \sin \frac{-2\pi}{3} \right) + \frac{18\sqrt{2}i}{2} = \\ &= \sqrt{2} \left( -\frac{1}{2} - \frac{\sqrt{3}}{2}i \right) + 9\sqrt{2}i = -\frac{\sqrt{2}}{2} + \left( 9\sqrt{2} - \frac{\sqrt{6}}{2} \right) i \end{aligned}$$

$$10.2 |z - z^3| = |z(1 - z^2)| = |z||1 - z^2| = |1 - e^{2i\alpha}| = \begin{cases} z = e^{i\alpha} \\ z^2 = e^{2i\alpha} \\ |z| = 1 \end{cases}$$

$$\begin{aligned} &= |1 - e^{2i\alpha}| = |1 - (\cos(2\alpha) + i \sin(2\alpha))| = \\ &= |(1 - \cos(2\alpha)) + i \sin(2\alpha)| = \\ &= \sqrt{(1 - \cos(2\alpha))^2 + (\sin(2\alpha))^2} = \\ &= \sqrt{1 - 2\cos(2\alpha) + \cos^2(2\alpha) + \sin^2(2\alpha)} = \\ &= \sqrt{1 - 2\cos(2\alpha) + 1} = \sqrt{2 - 2\cos(2\alpha)} = \\ &= \sqrt{2 - 2(\cos^2 \alpha - \sin^2 \alpha)} = \sqrt{2 - 2(1 - \sin^2 \alpha - \sin^2 \alpha)} = \\ &= \sqrt{2 - 2 + 4\sin^2 \alpha} = \sqrt{4\sin^2 \alpha} = 2|\sin \alpha| \end{aligned}$$

$$11.1 z_1 + z_2^3 = \sqrt{2} e^{-i\frac{3\pi}{4}} + (3 + 4i)^3 =$$

$$= \sqrt{2} \left( \cos \left( -\frac{3\pi}{4} \right) + i \sin \left( -\frac{3\pi}{4} \right) \right) + (3 + 4i)^2 (3 + 4i) =$$

$$= \sqrt{2} \left( -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i \right) + (9 + 24i + 16i^2)(3 + 4i) =$$

$$\begin{aligned} &= -1 - i + (-7 + 24i)(3 + 4i) = \\ &= -1 - i + (-21 - 28i + 72i + 96i^2) = \\ &= -1 - i + (-117 + 44i) = -118 + 43i \end{aligned}$$

$$11.2 z^3 = z_1^2 \Leftrightarrow z^3 = \left( \sqrt{2} e^{-i\frac{3\pi}{4}} \right)^2 \Leftrightarrow z^3 = (\sqrt{2})^2 e^{-i\frac{3\pi}{2}} \Leftrightarrow z^3 = 2e^{-i\frac{3\pi}{2}}$$

$$\Leftrightarrow z = \sqrt[3]{2} e^{-i\frac{3\pi}{2}} \Leftrightarrow z = \sqrt[3]{2} e^{i\left(\frac{-3\pi}{6} + \frac{2k\pi}{3}\right)}, k = \{0, 1, 2\}$$

$$\Leftrightarrow z = \sqrt[3]{2} e^{-i\frac{\pi}{2}} \vee z = \sqrt[3]{2} e^{i\frac{\pi}{6}} \vee z = \sqrt[3]{2} e^{i\frac{5\pi}{6}}$$

$$12.1 w = \frac{z_2 \times i^{4n+3} - b}{\sqrt{2} e^{-i\frac{\pi}{4}}} = \frac{(1 - \sqrt{2}i) \times (i^4)^n \times i^3 - b}{\sqrt{2} \left( \cos \left( -\frac{\pi}{4} \right) + i \sin \left( -\frac{\pi}{4} \right) \right)} =$$

$$= \frac{(1 - \sqrt{2}i) \times i^3 - b}{\sqrt{2} \left( \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i \right)} = \frac{(1 - \sqrt{2}i)(-i) - b}{1 - i} =$$

$$= \frac{-i - \sqrt{2} - b}{1 - i} = \frac{-\sqrt{2} - b - i}{1 - i} = \frac{\sqrt{2} + b + i}{-1 + i} =$$

$$= \frac{(\sqrt{2} + b + i)(-1 - i)}{(-1 + i)(-1 - i)} = \frac{-\sqrt{2} - \sqrt{2}i - b - bi - i - i^2}{(-1)^2 - i^2} =$$

$$= \frac{-\sqrt{2} - \sqrt{2}i - b - bi - i + 1}{2} = \frac{-\sqrt{2} - b + 1 + (-\sqrt{2} - b - 1)i}{2}$$

$$= \frac{-\sqrt{2} - b + 1}{2} + \frac{-\sqrt{2} - b - 1}{2}i$$

$w$  é um número imaginário puro. Logo:

$$\frac{-\sqrt{2} + b + 1}{2} \wedge \frac{-\sqrt{2} - b - 1}{2} \neq 0 \Leftrightarrow -\sqrt{2} - b + 1 = 0 \wedge -\sqrt{2} - b - 1 \neq 0$$

$$\Leftrightarrow -\sqrt{2} + 1 = b \wedge -\sqrt{2} - 1 \neq b \Leftrightarrow b = -\sqrt{2} + 1$$

Portanto,  $b = -\sqrt{2} + 1$ .

$$12.2 \cos x - \sqrt{3} \sin x + yi = z_1 \Leftrightarrow \cos x - \sqrt{3} \sin x + yi = 1 - \sqrt{2}i \Leftrightarrow$$

$$\Leftrightarrow \cos x - \sqrt{3} \sin x = 1 \wedge y = -\sqrt{2} \Leftrightarrow$$

$$\Leftrightarrow \frac{1}{2} \cos x - \frac{\sqrt{3}}{2} \sin x = \frac{1}{2} \wedge y = -\sqrt{2} \Leftrightarrow$$

$$\Leftrightarrow \cos \left( \frac{\pi}{3} \right) \cos x - \sin \left( \frac{\pi}{3} \right) \sin x = \cos \left( \frac{\pi}{3} \right) \wedge y = -\sqrt{2} \Leftrightarrow$$

$$\Leftrightarrow \cos \left( \frac{\pi}{3} + x \right) = \cos \left( \frac{\pi}{3} \right) \wedge y = -\sqrt{2} \Leftrightarrow$$

$$\Leftrightarrow \left( \frac{\pi}{3} + x = \frac{\pi}{3} + 2k\pi \vee \frac{\pi}{3} + x = -\frac{\pi}{3} + 2k\pi, k \in \mathbb{Z} \right) \wedge y = -\sqrt{2} \Leftrightarrow$$

$$\Leftrightarrow \left( x = 2k\pi \vee x = -\frac{2\pi}{3} + 2k\pi, k \in \mathbb{Z} \right) \wedge y = -\sqrt{2}$$



O menor valor de  $n \in \mathbb{N}$ , obtém-se para  $k = 0$ , ou seja:

$$n = 3 + 6 \times 0 = 3$$

Logo,  $n = 3$ .

$$\begin{aligned} 15.3 \quad z &= \frac{\bar{w}_1}{w_1 + 1} = \frac{2e^{-\frac{2\pi}{3}i}}{2e^{\frac{2\pi}{3}i} + 1} = \frac{2e^{-\frac{2\pi}{3}i}}{2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) + 1} = \\ &= \frac{2e^{-\frac{2\pi}{3}i}}{2\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) + 1} = \frac{2e^{-\frac{2\pi}{3}i}}{-1 + \sqrt{3}i + 1} = \frac{2e^{-\frac{2\pi}{3}i}}{\sqrt{3}i} = \frac{2e^{-\frac{2\pi}{3}i}}{\sqrt{3}e^{i\frac{\pi}{2}}} = \\ &= \frac{2}{\sqrt{3}}e^{i\left(-\frac{2\pi}{3} - \frac{\pi}{2}\right)} = \frac{2\sqrt{3}}{3}e^{-\frac{7\pi}{6}i} \end{aligned}$$

13.  $w = e^{i\theta}$ , então  $wn = e^{in\theta}$  e  $w^{-n} = e^{-in\theta}$ , portanto:

$$\begin{aligned} \frac{w^n - w^{-n}}{2i} &= \frac{e^{in\theta} - e^{-in\theta}}{2i} = \\ &= \frac{\cos(n\theta) + i\sin(n\theta) - [\cos(-n\theta) + i\sin(-n\theta)]}{2i} = \\ &= \frac{\cos(n\theta) + i\sin(n\theta) - \cos(-n\theta) - i\sin(-n\theta)}{2i} = \\ &= \frac{\cos(n\theta) + i\sin(n\theta) - \cos(n\theta) + i\sin(n\theta)}{2} = \\ &= \frac{i\sin(n\theta) + i\sin(n\theta)}{2i} = \frac{2i\sin(n\theta)}{2i} = \sin(n\theta) \end{aligned}$$

$$\begin{aligned} 14.1 \quad z_1 &= \frac{\cos\alpha + i\sin(-\alpha)}{\sin(\alpha) + i\sin(-\alpha)} = \frac{\cos(-\alpha) + i\sin(-\alpha)}{\sin(\alpha) + i\cos(\alpha)} = \\ &= \frac{e^{-i\alpha}}{\cos\left(\frac{\pi}{2} - \alpha\right) + i\sin\left(\frac{\pi}{2} - \alpha\right)} = \frac{e^{-i\alpha}}{e^{i\left(\frac{\pi}{2} - \alpha\right)}} = e^{i\left[-\alpha - \left(\frac{\pi}{2} - \alpha\right)\right]} = \\ &= e^{-i\frac{\pi}{2}} = \cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right) = 0 + i(-1) = -i \end{aligned}$$

Portanto,  $z_1$ , é um imaginário puro de coeficiente negativo, já que  $\text{Re}(z_1) = 0 \wedge \text{Im}(z_1) < 0$ .

$$\begin{aligned} 14.2 \quad z_2 &= \frac{\left[i^{4n+1} \times \left(e^{i\frac{\pi}{5}}\right)^5 - 1\right]^2}{2i^{10}} = \frac{(i^{4n} \times i \times e^{i\pi} - 1)^2}{2i^{2 \times 4 + 10} \times i^2} = \frac{(ix e^{i\pi} - 1)^2}{2 \times (i^4)^2 \times i^2} \\ &= \frac{(-i-1)^2}{2 \times 1 \times (-1)} = \frac{i^2 + 2i + 1}{-2} = \frac{2i}{-2} = -i, \text{ logo } z_2 = z_1. \end{aligned}$$

$$\begin{aligned} 15.1 \quad w_3 &= \frac{i}{1-i} - i^{24n+2} = \frac{i}{1-i} - (i^4)^{6n} \times i^2 = \frac{i}{1-i} - i^2 = \frac{i}{1-i} + 1 = \\ &= \frac{i+1-i}{1-i} = \frac{1}{1-i} = \frac{1+i}{(1-i)(1+i)} = \begin{cases} z = 1+i & \theta = \text{Arg}(z) \\ |z| = \sqrt{1+1} = \sqrt{2} \\ \tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4} \\ \theta \in 1.^\circ Q \\ z = \sqrt{2}e^{i\frac{\pi}{4}} \end{cases} \\ &= \frac{1}{2}(1+i) = \frac{1}{2} \times \sqrt{2}e^{i\frac{\pi}{4}} = \frac{\sqrt{2}}{2}e^{i\frac{\pi}{4}} \end{aligned}$$

$$\begin{aligned} 15.2 \quad (i^3 w_2)^n + 1 = 0 &\Leftrightarrow (-i w_2)^n = -1 \Leftrightarrow \left(-ie^{i\frac{5\pi}{6}}\right)^n = -1 \\ &\Leftrightarrow \left(e^{-i\frac{\pi}{2}} e^{i\frac{5\pi}{6}}\right)^n = -1 \Leftrightarrow \left(e^{i\left(-\frac{\pi}{2} + \frac{5\pi}{6}\right)}\right)^n = -1 \Leftrightarrow \left(e^{i\frac{\pi}{3}}\right)^n = -1 \\ &\Leftrightarrow e^{i\frac{\pi n}{3}} = e^{-i\pi} \Leftrightarrow \frac{\pi n}{3} = \pi + 2k\pi, k \in \mathbb{Z} \Leftrightarrow n = 3 + 6k, k \in \mathbb{Z} \end{aligned}$$

$$16.1 \quad z^4 + zz_1 = 0 \Leftrightarrow z^4 + z(\sqrt{3} + i) = 0$$

$$\Leftrightarrow z(z^3 + \sqrt{3} + i) = 0 \Leftrightarrow$$

$$\Leftrightarrow z = 0 \vee z^3 + \sqrt{3} + i = 0 \Leftrightarrow$$

$$\Leftrightarrow z = 0 \vee z^3 = -\sqrt{3} - i \Leftrightarrow$$

$$\Leftrightarrow z = 0 \vee z = \sqrt[3]{-\sqrt{3} - i} \Leftrightarrow z = 0 \vee z = \sqrt[3]{2e^{i\frac{7\pi}{6}}} \Leftrightarrow$$

$$\Leftrightarrow z = 0 \vee z = \sqrt[3]{2}e^{i\left(\frac{7\pi}{18} + \frac{2k\pi}{3}\right)}, k = 0, 1, 2$$

$$\Leftrightarrow z = 0 \vee z = \sqrt[3]{2}e^{i\frac{7\pi}{18}} \vee z = \sqrt[3]{2}e^{i\frac{19\pi}{18}} \vee z = \sqrt[3]{2}e^{i\frac{31\pi}{18}}$$

$$16.2 \quad |z_2 + z_3|^2 = \left|e^{i\frac{\pi}{5}} + 3 + i\right|^2 = \left|\cos\frac{\pi}{5} + i\sin\frac{\pi}{5} + 3 + i\right|^2 =$$

$$= \left|\left(\cos\frac{\pi}{5} + 3\right) + \left(\sin\frac{\pi}{5} + 1\right)i\right|^2 =$$

$$= \left[\left(\cos\frac{\pi}{5} + 3\right)^2 + \left(\sin\frac{\pi}{5} + 1\right)^2\right]^2 =$$

$$= \left(\cos\frac{\pi}{5} + 3\right)^2 + \left(\sin\frac{\pi}{5} + 1\right)^2 =$$

$$= \cos^2\frac{\pi}{5} + 6\cos\frac{\pi}{5} + 9 + \sin^2\frac{\pi}{5} + 2\sin\frac{\pi}{5} + 1 =$$

$$= \cos^2\frac{\pi}{5} + \sin^2\frac{\pi}{5} + 6\cos\frac{\pi}{5} + 2\sin\frac{\pi}{5} + 10 =$$

$$= 1 + 6\cos\frac{\pi}{5} + 2\sin\frac{\pi}{5} + 10 = 11 + 6\cos\frac{\pi}{5} + 2\sin\frac{\pi}{5}$$



**16.3**  $w = \frac{i - (z_2)^{10}}{z_1 - i^5} = \frac{i - \left(e^{i\frac{\pi}{5}}\right)^{10}}{\sqrt{3} + i - i^4 \times i} = \frac{i - e^{i2\pi}}{\sqrt{3} + i - i} = \frac{i - 1}{\sqrt{3}}$

$= \frac{i-1}{\sqrt{3}} = \frac{\sqrt{2} e^{i\frac{3\pi}{4}}}{\sqrt{3} e^{i\pi}} = \frac{\sqrt{2}}{\sqrt{3}} e^{i\left(\frac{3\pi}{4} - \pi\right)} = \frac{\sqrt{6}}{3} e^{i\frac{3\pi}{4}}$

$\left\{ \begin{array}{l} z = -1 + i \text{ e } \theta = \text{Arg}(z) \\ |z| = \sqrt{1+1} = \sqrt{2} \\ \tan \theta = -1 \Rightarrow \theta = \frac{3\pi}{4} \\ \theta \in 2.^\circ Q \\ z = \sqrt{2} e^{i\frac{3\pi}{4}} \end{array} \right.$

**16.4**  $z_1 = \sqrt{3} + i = 2e^{i\frac{\pi}{6}}$  e  $z_2 = e^{i\frac{\pi}{5}}$

$(z_1 \times z_2)^n = \left(2e^{i\frac{\pi}{6}} \times e^{i\frac{\pi}{5}}\right)^n = \left(2e^{i\left(\frac{\pi}{6} + \frac{\pi}{5}\right)}\right)^n$

$= \left(2e^{i\frac{11\pi}{30}}\right)^n = 2^n e^{i\frac{11\pi}{30}n}$

$\left\{ \begin{array}{l} z = \sqrt{3} + i \text{ e } \theta = \text{Arg}(z) \\ |z| = \sqrt{3+1} = 2 \\ \tan \theta = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \Rightarrow \theta = \frac{\pi}{6} \\ \theta \in 1.^\circ Q \\ \sqrt{3} + i = 2e^{i\frac{\pi}{6}} \end{array} \right.$

$\left(2^n \times i^{\frac{11\pi}{30}n}\right) \in \mathbb{R}^+ \Leftrightarrow \frac{11\pi}{30}n = 2k\pi, k \in \mathbb{Z} \Leftrightarrow$

$\Leftrightarrow n = \frac{30}{11\pi} \times 2k\pi, k \in \mathbb{Z} \Leftrightarrow n = \frac{60}{11}k, k \in \mathbb{Z}$

Portanto, o menor valor de  $n$  natural para o qual  $(z_1 \times z_2)^n$  é um número real positivo,  $n = 60$ , é obtido para  $k = 11$ .

$z^3 - z^2i + 16z - 16i = 0 \Leftrightarrow (z-i)(z^2 + 16) = 0 \Leftrightarrow$

$\Leftrightarrow z - i = 0 \vee z^2 + 16 = 0 \Leftrightarrow z = i \vee z^2 = -16 \Leftrightarrow$

$\Leftrightarrow z = i \vee z = -\sqrt{-16} \vee z = \sqrt{-16} \Leftrightarrow$

$\Leftrightarrow z = i \vee z = -4i \vee z = 4i$

**17.1** Pretende-se determinar  $w = x + iy$ , tal que  $w^2 = 7 + 24i$ .

$(z + iy)^2 = 7 + 24i \Leftrightarrow x^2 + 2xyi + i^2 y^2 = 7 + 24i$

$\Leftrightarrow (x^2 - y^2) + 2xyi = 7 + 24i \Leftrightarrow \begin{cases} x^2 - y^2 = 7 \\ 2xy = 24 \end{cases} \Leftrightarrow$

$\Leftrightarrow \begin{cases} x^2 - \left(\frac{12}{x}\right)^2 = 7 \\ y = \frac{12}{x} \end{cases} \Leftrightarrow \begin{cases} x^2 - \frac{144}{x^2} = 7 \\ y = \frac{12}{x} \end{cases} \Leftrightarrow \begin{cases} x^4 - 7x^2 - 144 = 0 \\ y = \frac{12}{x} \end{cases}$

$\Leftrightarrow \begin{cases} x^2 = \frac{7 \pm \sqrt{(-7)^2 - 4 \times (-144)}}{2} \\ y = \frac{12}{x} \end{cases} \Leftrightarrow \begin{cases} x^2 = \frac{7 \pm 25}{2} \\ y = \frac{12}{x} \end{cases} \Leftrightarrow$

$\Leftrightarrow \begin{cases} x^2 = 16 \vee x^2 = -9 \\ y = \frac{12}{x} \end{cases} \Leftrightarrow \begin{cases} x^2 = 16 \vee x \in \emptyset \\ y = \frac{12}{x} \end{cases} \Leftrightarrow \begin{cases} x^2 = 16 \\ y = \frac{12}{x} \end{cases}$

$\Leftrightarrow \begin{cases} x = 4 \\ y = 3 \end{cases} \vee \begin{cases} x = -4 \\ y = -3 \end{cases}$

**17.2** As raízes quadradas de  $z_1 = 7 + 24i$  são  $4 + 3i$  e  $-4 - 3i$ .  $z^3 - z^2i + 16z - 16i = 0 \Leftrightarrow z^3 - z^2 + 16z - 16i = 0$

Como  $z_2 = i$  é a solução da equação, podemos recorrer à regra de Ruffini:

	1	-i	16	-16i
i		i	0	16i
	1	0	16	0

**18.1** Seja  $z_1 = |z_1|e^{i\theta}$ , com  $\theta \in \left[\frac{\pi}{2}, \pi\right]$ . Então,  $z^3 = |z_0|^3 e^{i3\theta}$ .

Se  $\frac{\pi}{2} < \theta < \pi$ , então  $\frac{3\pi}{2} < 3\theta < 3\pi$ . Se  $3\theta \in \left[\frac{3\pi}{2}, 3\pi\right]$ , o

**18.2** afixo de  $z^3$  não pode pertencer ao terceiro quadrante.

$z_3 \times \bar{z}_3 = |z_3|^2 = (2\sqrt{3})^2 = 12$

$\frac{z_3 \times \bar{z}_3}{8} = \frac{12}{8} = \frac{3}{2}$

Como  $z_2 = |z_2|e^{i\frac{\pi}{3}}$ ,  $\frac{z_2}{|z_2|} = \frac{|z_2|e^{i\frac{\pi}{3}}}{|z_2|} = e^{i\frac{\pi}{3}}$  e

$\left(\frac{z_2}{|z_2|}\right)^9 = \left(e^{i\frac{\pi}{3}}\right)^9 = \left(e^{i\frac{9\pi}{3}}\right) = e^{i3\pi} = e^{i\pi} = -1$

Portanto,  $\frac{z_3 \times \bar{z}_3}{8} + \left(\frac{z_2}{|z_2|}\right)^9 = \frac{3}{2} + (-1) = \frac{1}{2}$ .

**19.1**  $(z_1)^3 = zi + 2z \Leftrightarrow (1 + 2i)^3 = zi + 2z$

$\Leftrightarrow (1 + 2i^2)(1 + 2i) = z(i + 2) \Leftrightarrow$

$\Leftrightarrow (1 + 4i + 4i^2)(1 + 2i) = z(i + 2) \Leftrightarrow$

$\Leftrightarrow (-3 + 4i)(1 + 2i) = z(i + 2) \Leftrightarrow$

$\Leftrightarrow -3 - 6i + 4i + 8i^2 = z(i + 2) \Leftrightarrow$

$\Leftrightarrow -11 - 2i = z(i + 2) \Leftrightarrow z = \frac{-11 - 2i}{i + 2} \Leftrightarrow$

$\Leftrightarrow z = \frac{(-11 - 2i)(2 - i)}{(2 + i)(2 - i)} \Leftrightarrow z = \frac{-22 + 11i - 4i + 2i^2}{4 + 1} \Leftrightarrow$

$\Leftrightarrow z = \frac{-24 + 7i}{5} \Leftrightarrow z = -\frac{24}{5} + \frac{7}{5}i$

**19.2** A solução da equação é  $z = -\frac{24}{5} + \frac{7}{5}i$ .

Dado que  $z_2$  é uma das raízes cúbicas de  $w$ :

$w = (z_2)^2 = (4 - \sqrt{3}i + i^{4n+2018})^3 = (4 - \sqrt{3}i + i^{4n} \times i^{2018})^3$



$$\begin{aligned}
 &= (4 - \sqrt{3}i + i^{2018})^3 = (4 - \sqrt{3}i + i^2)^3 = (4 - \sqrt{3}i - 1)^3 = \\
 &= (3 - \sqrt{3}i)^3 = (3 - \sqrt{3}i)^2 (3 - \sqrt{3}i) = \left| \begin{array}{l} 2018 \lfloor 4 \\ 2 \quad 504 \end{array} \right. \\
 &= (9 - 6\sqrt{3}i + 3i^2)(3 - \sqrt{3}i) = (6 - 6\sqrt{3}i)(3 - \sqrt{3}i) = \\
 &= 18 - 6\sqrt{3}i + 18i^2 = -24\sqrt{3}i
 \end{aligned}$$

**20.** O ponto  $D$  obtém-se do ponto  $A$  por uma reflexão de eixo  $Ox$ . Portanto, o número complexo cujo afixo é o ponto  $D$  é o conjugado de  $z$ , ou seja,  $\bar{z} = \rho e^{-i\theta}$ .

O ponto  $B$  obtém-se do ponto  $D$  por uma reflexão central de centro  $O$ . Logo, o número complexo cujo afixo é o ponto  $B$  é o simétrico de  $\bar{z} = \rho e^{-i\theta}$ , ou seja,  $-\bar{z} = \rho e^{-i(\theta-\pi)} = \rho e^{i(\pi-\theta)}$ .

Finalmente, o ponto  $C$  obtém-se do ponto  $A$  por uma reflexão central de centro  $O$ . O número complexo cujo afixo é o ponto  $C$  é o simétrico de  $z$ , ou seja,  $-z = \rho e^{i(\pi+\theta)}$ .

**21.1**  $|z| \times z^2 = z^3 \Leftrightarrow |z| \times (|z|e^{i\theta})^2 = (\sqrt{3}-i)^3 \Leftrightarrow$

$$\begin{aligned}
 &\Leftrightarrow |z| \times |z|^2 e^{i2\theta} = \left( 2e^{-i\frac{\pi}{6}} \right)^3 \Leftrightarrow \begin{cases} z = \sqrt{3}-i \text{ e } \theta = \text{Arg}(z) \\ |z| = \sqrt{3+1} = 2 \\ \tan \theta = \frac{-1}{\sqrt{3}} = -\frac{\sqrt{3}}{3} \Rightarrow \theta = -\frac{\pi}{6} \\ \theta \in 4.^\circ Q \\ \sqrt{3}-i = 2e^{-i\frac{\pi}{6}} \end{cases} \\
 &\Leftrightarrow |z|^3 e^{i2\theta} = 2^3 e^{-i\frac{3\pi}{6}} \Leftrightarrow \\
 &\Leftrightarrow |z|^3 e^{i2\theta} = 8e^{-i\frac{\pi}{2}} \Leftrightarrow \\
 &\Leftrightarrow |z|^3 = 8 \wedge 2\theta = -\frac{\pi}{2} + 2k\pi, k \in \mathbb{Z} \Leftrightarrow \\
 &\Leftrightarrow z = 2e^{-i\frac{\pi}{4}} \vee z = 2e^{i\frac{3\pi}{4}}
 \end{aligned}$$

**21.2**  $w$  é o inverso de  $z_1 \times z_2$ .

$$\begin{aligned}
 w &= \frac{1}{z_1 \times z_2} = \frac{1}{(\sqrt{3}-i)(-2i)} = \\
 &= \frac{1}{2e^{-i\frac{\pi}{6}} \times 2e^{-i\frac{\pi}{2}}} = \frac{1}{4e^{i\left(\frac{\pi}{6} + \frac{\pi}{2}\right)}} = \\
 &= \frac{e^{i\pi}}{4e^{i\frac{2\pi}{3}}} = \frac{1}{4}e^{i\left(\pi - \frac{2\pi}{3}\right)} = \frac{1}{4}e^{i\frac{2\pi}{3}} \quad \begin{cases} z = \sqrt{3}-i \text{ e } \theta = \text{Arg}(z) \\ |z| = \sqrt{3+1} = 2 \\ \tan \theta = \frac{-1}{\sqrt{3}} = -\frac{\sqrt{3}}{3} \Rightarrow \theta = -\frac{\pi}{6} \\ \theta \in 4.^\circ Q \\ \sqrt{3}-i = 2e^{-i\frac{\pi}{6}} \\ -2i = 2e^{-i\frac{\pi}{2}} \end{cases}
 \end{aligned}$$

**22.**  $\frac{w}{z_1} = 2 + 2i \Leftrightarrow \frac{w}{1 + \sqrt{3}i} = 2 + 2i \Leftrightarrow w = (2 + 2i) \times (1 + \sqrt{3}i)$

$$\begin{aligned}
 &\Leftrightarrow w = 2\sqrt{2}e^{i\frac{\pi}{4}} \times 2e^{i\frac{\pi}{3}} \Leftrightarrow \begin{cases} z_1 = 2 + 2i = |z_1|e^{i\theta} \\ |z_1| = \sqrt{4+4} = 2\sqrt{2} \\ \theta_1 = \arctan \frac{2}{2} = \frac{\pi}{4} \\ z_2 = 1 + \sqrt{3}i = |z_2|e^{i\theta} \\ |z_2| = \sqrt{1+3} = 2 \\ \theta_2 = \arctan \sqrt{3} = \frac{\pi}{3} \end{cases} \\
 &\Leftrightarrow w = 4\sqrt{2}e^{i\left(\frac{\pi}{4} + \frac{\pi}{3}\right)} \Leftrightarrow w = 4\sqrt{2}e^{i\frac{7\pi}{12}} \\
 &|w| = 4\sqrt{2} \text{ e } \text{Arg } w = \frac{7\pi}{12}
 \end{aligned}$$

**23.** Seja  $z = x + iy$ , então,  $\bar{z} = x - iy$ . Substituindo, na equação:

$$\begin{aligned}
 &\frac{2(x+iy) + 3i(x-iy+2)}{1+i} = 13 + 4i \Leftrightarrow \\
 &\Leftrightarrow 2x + 2iy + 3ix + 3y + 6i = 13 + 13i + 4i + 4i^2 \Leftrightarrow \\
 &\Leftrightarrow (2x + 3y) + (3x + 2y)i + 6i = 9 + 17i \Leftrightarrow \\
 &\Leftrightarrow (2x + 3y) + (3x + 2y)i = 9 + 11i \Leftrightarrow \begin{cases} 2x + 3y = 9 \\ 3x + 2y = 11 \end{cases} \\
 &\Leftrightarrow \begin{cases} x = \frac{9-3y}{2} \\ 3 \times \frac{9-3y}{2} + 2y = 11 \end{cases} \Leftrightarrow \begin{cases} x = \frac{9-3y}{2} \\ 27-9y+4y = 22 \end{cases} \Leftrightarrow \begin{cases} x = 3 \\ y = 1 \end{cases}
 \end{aligned}$$

Logo,  $z = 3 + i$  é a solução da equação.

**24.**  $z + 2i = iz + k \Leftrightarrow z - iz = k - 2i \Leftrightarrow z = \frac{k-2i}{1-i}$

$$\begin{aligned}
 \frac{w}{z} &= 2 + 2i \Leftrightarrow w = z(2 + 2i) \Leftrightarrow z = \frac{k-2i}{1-i} \\
 &\Leftrightarrow w = \left( \frac{k-2i}{1-i} \right) (2 + 2i) \Leftrightarrow w = (k-2i) \left( \frac{2+2i}{1-i} \right) \Leftrightarrow \\
 &\Leftrightarrow w = (k-2i) \times \frac{(2+2i)(1+i)}{(1-i)(1+i)} \Leftrightarrow \\
 &\Leftrightarrow w = (k-2i) \times \frac{2+4i-2}{2} \Leftrightarrow w = (k-2i)2i \Leftrightarrow \\
 &\Leftrightarrow w = 2ki - 4i^2 \Leftrightarrow w = 4 + 2ki
 \end{aligned}$$

Como  $\text{Im}(z) = 8$ , tem-se que  $2k = 8$ , ou seja,  $k = 4$ .

**25.**  $|z+w|^2 - |z-\bar{w}|^2 = (z+w) \times (\bar{z}+\bar{w}) - (z-\bar{w}) \times (\bar{z}-\bar{w}) =$

$$\begin{aligned}
 &= (z+w) \times (\bar{z}+\bar{w}) - (z-\bar{w}) \times (\bar{z}-\bar{w}) = \\
 &= z\bar{z} + z\bar{w} + w\bar{z} + w\bar{w} - z\bar{z} + z\bar{w} + \bar{w}\bar{z} - \bar{w}\bar{w} = \\
 &= (z\bar{w} + z\bar{w}) + (w\bar{z} + \bar{w}\bar{z}) = z(\bar{w} + \bar{w}) + \bar{z}(w + \bar{w}) = \\
 &= (z + \bar{z})(\bar{w} + w) = 2 \times \frac{z + \bar{z}}{2} \times 2 \times \frac{\bar{w} + w}{2} \quad \begin{cases} \frac{z + \bar{z}}{2} = \text{Re}(z) \\ \frac{\bar{w} + w}{2} = \text{Re}(w) \end{cases} \\
 &= 4 \times \text{Re}(z) \times \text{Re}(w)
 \end{aligned}$$

**26.1**  $z = \frac{1+i}{1-i} = \frac{\sqrt{2}e^{i\frac{\pi}{4}}}{\sqrt{2}e^{-i\frac{\pi}{4}}} = e^{i\left(\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right)} = e^{i\frac{\pi}{2}}$

$$\begin{aligned}
 w &= \frac{\sqrt{2}}{1-i} = \frac{\sqrt{2}e^{i\pi/4}}{\sqrt{2}e^{-i\pi/4}} = e^{i\left(\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right)} = e^{i\frac{\pi}{2}} \\
 &= \frac{\sqrt{2}}{1-i} = \frac{\sqrt{2}e^{i\pi/4}}{\sqrt{2}e^{-i\pi/4}} = e^{i\left(\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right)} = e^{i\frac{\pi}{2}} \\
 &= \frac{\sqrt{2}}{1-i} = \frac{\sqrt{2}e^{i\pi/4}}{\sqrt{2}e^{-i\pi/4}} = e^{i\left(\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right)} = e^{i\frac{\pi}{2}}
 \end{aligned}$$

$$\begin{cases} z = 1+i \text{ e } \theta = \text{Arg}(z) \\ |z| = \sqrt{1+1} = \sqrt{2} \\ \tan \theta = 1 \Rightarrow \theta = \frac{\pi}{4} \\ \theta \in 1.^\circ Q \\ z = \sqrt{2}e^{i\frac{\pi}{4}} \\ 1-i = 1+i = \sqrt{2}e^{-i\frac{\pi}{4}} \end{cases}$$

**26.2** O quadrilátero  $[OABC]$  é um paralelogramo.

$$A\hat{O}C = \text{Arg}(z) - \text{Arg}(w) = \left(\frac{\pi}{2} - \frac{\pi}{4}\right) \text{rad} = \frac{\pi}{4} \text{ rad}$$

$$A\hat{O}B = B\hat{O}C = \frac{\pi}{4} \text{ rad} = \frac{\pi}{8} \text{ rad}$$

$$\text{Arg}(z+w) = \left(\frac{\pi}{2} + \frac{\pi}{8}\right) \text{rad} = \frac{3\pi}{8} \text{ rad}$$

$$\begin{aligned}
 z+w &= \frac{1+i}{1-i} + \frac{\sqrt{2}}{1-i} = \frac{(1+\sqrt{2})+i}{1-i} = \frac{((1+\sqrt{2})+i)(1+i)}{(1-i)(1+i)} \\
 &= \frac{1+\sqrt{2}+(1+\sqrt{2})i+i+i^2}{1-i^2} = \frac{1+\sqrt{2}+(1+\sqrt{2}+1)i-1^2}{1-i^2} = \\
 &= \frac{\sqrt{2}+(2+\sqrt{2})i}{2} = \frac{\sqrt{2}}{2} + \frac{2+\sqrt{2}}{2}i
 \end{aligned}$$

$$\tan\left(\frac{3\pi}{8}\right) = \frac{\frac{2+\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = \frac{2+\sqrt{2}}{\sqrt{2}} = \frac{(2+\sqrt{2})\sqrt{2}}{(\sqrt{2})^2} = 1 + \sqrt{2}$$