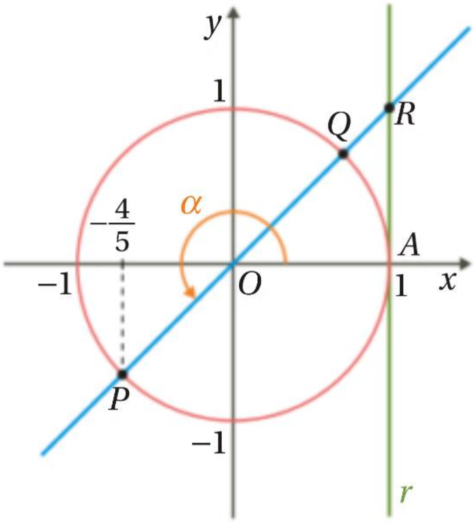
1. Na figura está representada a circunferência trigonométrica, num referencial o.n. Oxy.

Sabe-se que:

- o ponto A tem coordenadas $(1,0)$;
- a reta r é tangente à circunferência no ponto A;
- $[PQ]$ é um diâmetro da circunferência;
- o ponto R é a interseção da reta r com a reta PQ;
- o ângulo AOP, designado por α, pertence ao 3.º quadrante;
- o ponto P tem abcissa igual a $-\frac{4}{5}$.



Determine $\operatorname{sen}\alpha$ e $\tan\alpha$.

Sabemos que P tem abcissa $-\frac{4}{5}$, logo $\cos\alpha=-\frac{4}{5}$

$$\cos^2\alpha+\operatorname{sen}^2\alpha=1 \Leftrightarrow \left(-\frac{4}{5}\right)^2+\operatorname{sen}^2\alpha=1 \Leftrightarrow \operatorname{sen}^2\alpha=1-\frac{16}{25} \Leftrightarrow \operatorname{sen}\alpha=\pm\sqrt{\frac{9}{25}} \underset{\substack{\alpha\in 3.^{\circ}Q \\ \operatorname{sen}\alpha<0}}{\Leftrightarrow} \operatorname{sen}\alpha=-\frac{3}{5}$$

$$\tan\alpha=\frac{\operatorname{sen}\alpha}{\cos\alpha} \Leftrightarrow \tan\alpha=\frac{-\frac{3}{5}}{-\frac{4}{5}} \Leftrightarrow \tan\alpha=\frac{3}{4}$$

2. Sabendo que $\operatorname{sen}\alpha=\frac{1}{5}$ e α pertence ao 2.º quadrante, determine $\cos\alpha$ e $\tan\alpha$.

$$\operatorname{sen}^2\alpha+\cos^2\alpha=1 \Leftrightarrow \left(\frac{1}{5}\right)^2+\cos^2\alpha=1 \Leftrightarrow \cos^2\alpha=1-\frac{1}{25} \Leftrightarrow \cos\alpha=\pm\sqrt{\frac{24}{25}} \underset{\substack{\alpha\in\left]\frac{\pi}{2},\pi\right[\\ \cos\alpha<0}}{\Leftrightarrow} \cos\alpha=-\frac{\sqrt{24}}{5}$$

$$\tan\alpha=\frac{\operatorname{sen}\alpha}{\cos\alpha} \Leftrightarrow \tan\alpha=\frac{\frac{1}{5}}{-\frac{\sqrt{24}}{5}} \Leftrightarrow \tan\alpha=-\frac{1}{\sqrt{24}} \Leftrightarrow \tan\alpha=-\frac{\sqrt{6}}{12}$$

3. Sabendo que $\tan\alpha = -\frac{3}{2}$ e α pertence ao 4.º quadrante, determine $\cos\alpha + \operatorname{sen}\alpha$.

$$\tan^2\alpha + 1 = \frac{1}{\cos^2\alpha} \Leftrightarrow \left(-\frac{3}{2}\right)^2 + 1 = \frac{1}{\cos^2\alpha} \Leftrightarrow \frac{9}{4} + 1 = \frac{1}{\cos^2\alpha} \Leftrightarrow \frac{13}{4} = \frac{1}{\cos^2\alpha} \Leftrightarrow \frac{4}{13} = \cos^2\alpha \Leftrightarrow$$

$$\Leftrightarrow \cos\alpha = \pm\sqrt{\frac{4}{13}} \underset{\substack{\alpha\in\left]\frac{3\pi}{2},2\pi\right[\\ \cos\alpha>0}}{\Leftrightarrow} \cos\alpha = \frac{2}{\sqrt{13}} \Leftrightarrow \cos\alpha = \frac{2\sqrt{13}}{13}$$

$$\tan\alpha = \frac{\operatorname{sen}\alpha}{\cos\alpha} \Leftrightarrow -\frac{3}{2} = \frac{\operatorname{sen}\alpha}{\frac{2\sqrt{13}}{13}} \Leftrightarrow \operatorname{sen}\alpha = -\frac{3}{2}\times\frac{2\sqrt{13}}{13} \Leftrightarrow \operatorname{sen}\alpha = -\frac{3\sqrt{13}}{13}$$

Assim, $\cos\alpha + \operatorname{sen}\alpha = \frac{2\sqrt{13}}{13} - \frac{3\sqrt{13}}{13} = -\frac{\sqrt{13}}{13}$

4. Sabendo que $\operatorname{sen}\alpha = -\frac{1}{3}$ e que $\alpha \in \left]\pi, \frac{3\pi}{2}\right[$, calcule o valor exato de $\cos\alpha$ e de $\tan\alpha$.

$$\sin^2\alpha + \cos^2\alpha = 1 \Leftrightarrow \cos^2\alpha = 1 - \left(-\frac{1}{3}\right)^2 \Leftrightarrow \cos^2\alpha = \frac{8}{9}$$

Como $\alpha \in 3.^\circ$ Q., tem-se: $\cos\alpha = -\sqrt{\frac{8}{9}} = -\frac{2\sqrt{2}}{3}$

$$\tan\alpha = \frac{-\frac{1}{3}}{-\frac{2\sqrt{2}}{3}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$$

5. Prove as seguintes igualdades para α, tal que $\cos\alpha \neq 0$, $\operatorname{sen}\alpha \neq 0$, $\cos\beta \neq 0$ e $\operatorname{sen}\beta \neq 0$.

5.1. $(\cos\beta - \operatorname{sen}\beta)^2 = 2 - (\cos\beta + \operatorname{sen}\beta)^2$

$$(\cos\beta - \sin\beta)^2 = \cos^2\beta - 2\cos\beta\sin\beta + \sin^2\beta =$$

$$= (\cos^2\beta + \sin^2\beta) - 2\cos\beta\sin\beta = 1 - 2\cos\beta\sin\beta$$

$$= 2 - (1 + 2\cos\beta\sin\beta) = 2 - (\cos^2\beta + 2\cos\beta\sin\beta + \sin^2\beta)$$

$$= 2 - (\cos\beta + \sin\beta)^2$$

5.2. $\dfrac{\cos^2\alpha}{1+\text{sen}^2\alpha}=1-\text{sen}\,\alpha$

$$\frac{\cos^2\alpha}{1+\sin\alpha}=\frac{1-\sin^2\alpha}{1+\sin\alpha}=\frac{(1-\sin\alpha)(1+\sin\alpha)}{1+\sin\alpha}=1-\sin\alpha$$

5.3. $\tan\alpha+\dfrac{1}{\tan\alpha}=\dfrac{1}{\text{sen}\,\alpha\times\cos\alpha}$

$$\tan\alpha+\frac{1}{\tan\alpha}=\frac{\text{sen}\,\alpha}{\cos\alpha}+\frac{1}{\frac{\text{sen}\,\alpha}{\cos\alpha}}=\frac{\text{sen}\,\alpha}{\cos\alpha}+\frac{\cos\alpha}{\text{sen}\,\alpha}=\frac{\text{sen}^2\alpha}{\cos\alpha\times\text{sen}\,\alpha}+\frac{\cos^2\alpha}{\text{sen}\,\alpha\times\cos\alpha}=$$

$$=\frac{\text{sen}^2\alpha+\cos^2 a}{\text{sen}\,\alpha\times\cos\alpha}=\frac{1}{\text{sen}\,\alpha\times\cos\alpha}\qquad \text{c.q.m.}$$

5.4. $\dfrac{\text{sen}\,\beta}{1+\cos\beta}=\dfrac{1-\cos\beta}{\text{sen}\,\beta}$

$$\frac{\sin\beta}{1+\cos\beta}=\frac{\sin\beta(1-\cos\beta)}{(1+\cos\beta)(1-\cos\beta)}=\frac{\sin\beta(1-\cos\beta)}{1-\cos^2\beta}=$$

$$=\frac{\sin\beta(1-\cos\beta)}{\sin^2\beta}=\frac{1-\cos\beta}{\sin\beta}$$

5.5. $\dfrac{1-\tan^2\alpha}{1+\tan^2\alpha}+\text{sen}^2\alpha=\cos^2\alpha$

$$\frac{1-\tan^2\alpha}{1+\tan^2\alpha}+\text{sen}^2\alpha=\frac{1-\left(\frac{\text{sen}\,\alpha}{\cos\alpha}\right)^2}{1+\left(\frac{\text{sen}\,\alpha}{\cos\alpha}\right)^2}+\text{sen}^2\alpha=\frac{1-\frac{\text{sen}^2\alpha}{\cos^2\alpha}}{1+\frac{\text{sen}^2\alpha}{\cos^2\alpha}}+\text{sen}^2\alpha=\frac{\frac{\cos^2\alpha-\text{sen}^2\alpha}{\cancel{\cos^2\alpha}}}{\frac{\cos^2\alpha+\text{sen}^2\alpha}{\cancel{\cos^2\alpha}}}+\text{sen}^2\alpha=$$

$$=\frac{\cos^2\alpha-\text{sen}^2\alpha}{1}+\text{sen}^2\alpha=\cos^2\alpha\qquad \text{c.q.m.}$$

5.6. $\text{sen}\,\alpha\times\cos\alpha\times\left(\tan\alpha+\dfrac{1}{\tan\alpha}\right)=1$

$$\text{sen}\,\alpha\times\cos\alpha\times\left(\tan\alpha+\frac{1}{\tan\alpha}\right)=\text{sen}\,\alpha\cos\alpha\left(\frac{\text{sen}\,\alpha}{\cos\alpha}+\frac{1}{\frac{\text{sen}\,\alpha}{\cos\alpha}}\right)=\text{sen}\,\alpha\cos\alpha\left(\frac{\text{sen}\,\alpha}{\cos\alpha}+\frac{\cos\alpha}{\text{sen}\,\alpha}\right)=$$

$$=\cancel{\text{sen}\,\alpha\cos\alpha}\left(\frac{\text{sen}^2\alpha+\cos^2\alpha}{\cancel{\text{sen}\,\alpha\cos\alpha}}\right)=\text{sen}^2\alpha+\cos^2\alpha=1\qquad \text{c.q.m.}$$

5.7. $\left(2-\frac{1}{\cos^2\alpha}\right)\left(1-\text{sen}^2\alpha\right)=2\cos^2\alpha-1$

$$\left(2-\frac{1}{\cos^2\alpha}\right)\left(1-\text{sen}^2\alpha\right)=\left(\frac{2\cos^2\alpha-1}{\cos^2\alpha}\right)\cos^2\alpha=2\cos^2\alpha-1 \quad \text{c.q.m.}$$

6. Sendo x um ângulo agudo, mostre que:

6.1. $\frac{\cos^3 x-\cos x}{\text{sen}^3 x-\text{sen}\,x}=\tan x$

$$\frac{\cos^3 x-\cos x}{\text{sen}^3 x-\text{sen}\,x}=\frac{\cos x\left(\cos^2 x-1\right)}{\text{sen}\,x\left(\text{sen}^2 x-1\right)}=\frac{\cos x\left(-\text{sen}^2 x\right)}{\text{sen}\,x\left(-\cos^2 x\right)}=\frac{\text{sen}\,x}{\cos x}=\tan x \quad \text{c.q.m.}$$

6.2. $\frac{\tan^2 x\times\cos x}{1+\frac{1}{\tan^2 x}}=\tan x\times\text{sen}^3 x$

$$\frac{\tan^2 x\times\cos x}{1+\frac{1}{\tan^2 x}}=\frac{\frac{\text{sen}^2 x}{\cos^2 x}\times\cos x}{1+\frac{1}{\frac{\text{sen}^2 x}{\cos^2 x}}}=\frac{\frac{\text{sen}^2 x}{\cos x}}{1+\frac{\cos^2 x}{\text{sen}^2 x}}=\frac{\frac{\text{sen}^2 x}{\cos x}}{\frac{\text{sen}^2 x+\cos^2 x}{\text{sen}^2 x}}=\frac{\frac{\text{sen}^2 x}{\cos x}}{\frac{1}{\text{sen}^2 x}}=\frac{\text{sen}^4 x}{\cos x}=$$

$$=\frac{\text{sen}\,x}{\cos x}\times\text{sen}^3 x=\tan x\times\text{sen}^3 x \quad \text{c.q.m.}$$

6.3. $\left(\tan^3 x+\tan x\right)\cos^3 x=\text{sen}\,x$

$$\left(\tan^3 x+\tan x\right)\cos^3 x=\left(\frac{\text{sen}^3 x}{\cos^3 x}+\frac{\text{sen}\,x}{\cos x}\right)\cos^3 x=\frac{\text{sen}^3 x\times\cos^3 x}{\cos^3 x}+\frac{\text{sen}\,x\times\cos^{3\,2} x}{\cos x}=$$

$$=\text{sen}^3 x+\text{sen}\,x\cos^2 x=\text{sen}\,x\left(\text{sen}^2 x+\cos^2 x\right)=\text{sen}\,x \quad \text{c.q.m.}$$

6.4. $\frac{\text{sen}^4 x-\text{sen}^2 x}{\frac{1}{\cos^2 x}-1}+\cos^4 x=0$

$$\frac{\text{sen}^4 x-\text{sen}^2 x}{\frac{1}{\cos^2 x}-1}+\cos^4 x=\frac{\text{sen}^2 x\left(\text{sen}^2 x-1\right)}{\frac{1-\cos^2}{\cos^2 x}}+\cos^4 x=\frac{\text{sen}^2 x\left(-\cos^2 x\right)}{\frac{\text{sen}^2 x}{\cos^2 x}}+\cos^4 x=$$

$$=\frac{-\text{sen}^2 x\cos^4 x}{\text{sen}^2 x}+\cos^4 x=-\cos^4 x+\cos^4 x=0 \quad \text{c.q.m.}$$

6.5. $2\,\text{sen}^2 x \cos^2 x = 1 - \text{sen}^4 x - \cos^4 x$

$$1 - \text{sen}^4 x - \cos^4 x = 1 - \left(\text{sen}^4 x + \cos^4 x\right) =$$

$$= 1 - \left(\underbrace{\text{sen}^4 x + 2\,\text{sen}^2 x \cos^2 x + \cos^4 x}_{\left(\text{sen}^2 x + \cos^2 x\right)^2} - 2\,\text{sen}^2 x \cos^2 x\right) =$$

$$= 1 - \left(\text{sen}^2 x + \cos^2 x\right)^2 + 2\,\text{sen}^2 x \cos^2 x = 1 - 1^2 + 2\,\text{sen}^2 x \cos^2 x =$$

$$= 2\,\text{sen}^2 x \cos^2 x \qquad \text{c.q.m.}$$

6.6. $\dfrac{\text{sen}\, x}{1 - \cos x} = \dfrac{1}{\text{sen}\, x} + \dfrac{1}{\tan x}$

$$\frac{\text{sen}\, x}{1 - \cos x} = \frac{\text{sen}\, x\,(1 + \cos x)}{(1 - \cos x)(1 + \cos x)} = \frac{\text{sen}\, x\,(1 + \cos x)}{1 - \cos^2 x} = \frac{\cancel{\text{sen}\, x}\,(1 + \cos x)}{\text{sen}^{\cancel{2}} x} = \frac{1 + \cos x}{\text{sen}\, x} =$$

$$= \frac{1}{\text{sen}\, x} + \frac{\cos x}{\text{sen}\, x} = \frac{1}{\text{sen}\, x} + \frac{1}{\frac{\text{sen}\, x}{\cos x}} = \frac{1}{\text{sen}\, x} + \frac{1}{\tan x} \qquad \text{c.q.m.}$$

6.7. $\cos^4 x - \text{sen}^4 x = 1 - 2\,\text{sen}^2 x$

$$\cos^4 x - \text{sen}^4 x = \left(\cos^2 x\right)^2 - \left(\text{sen}^2 x\right)^2 = \underbrace{\left(\cos^2 x + \text{sen}^2 x\right)}_{1}\left(\cos^2 x - \text{sen}^2 x\right) =$$

$$= 1 - \text{sen}^2 x - \text{sen}^2 x = 1 - 2\,\text{sen}^2 x \qquad \text{c.q.m.}$$

6.8. $\dfrac{1}{\text{sen}^2 x} - 1 = \dfrac{1}{\tan^2 x}$

$$\frac{1}{\text{sen}^2 x} - 1 = \frac{1 - \text{sen}^2 x}{\text{sen}^2 x} = \frac{\cos^2 x}{\text{sen}^2 x} = \frac{1}{\frac{\text{sen}^2 x}{\cos^2 x}} = \frac{1}{\tan^2 x} \qquad \text{c.q.m.}$$

6.9. $\dfrac{1 - \text{sen}\, x}{\cos x} + \dfrac{\cos x}{1 + \text{sen}\, x} = \dfrac{2\cos x}{1 + \text{sen}\, x}$, com $\cos x \neq 0$ e $\text{sen}\, x \neq -1$

$$\frac{1 - \text{sen}\, x}{\cos x} + \frac{\cos x}{1 + \text{sen}\, x} = \frac{(1 - \text{sen}\, x)(1 + \text{sen}\, x) + \cos^2 x}{\cos x\,(1 + \text{sen}\, x)} = \frac{1 - \text{sen}^2 x + \cos^2 x}{\cos x\,(1 + \text{sen}\, x)} = \frac{\cos^2 x + \cos^2 x}{\cos x\,(1 + \text{sen}\, x)} =$$

$$= \frac{2\cos^{\cancel{2}} x}{\cancel{\cos x}\,(1 + \text{sen}\, x)} = \frac{2\cos x}{1 + \text{sen}\, x} \qquad \text{c.q.m.}$$

6.10. $\text{sen}\, x \tan x + \cos x = \dfrac{1}{\cos x}$, com $\cos x \neq 0$

$$\text{sen}\, x \tan x + \cos x = \text{sen}\, x \frac{\text{sen}\, x}{\cos x} + \cos x = \frac{\text{sen}^2 x + \cos^2 x}{\cos x} = \frac{1}{\cos x} \qquad \text{c.q.m.}$$

6.11. $(\operatorname{sen} x+\tan x)(1-\cos x)=\dfrac{\operatorname{sen}^3 x}{\cos x}$, com $\cos x \neq 0$

$$(\operatorname{sen} x+\tan x)(1-\cos x)=\operatorname{sen} x-\operatorname{sen} x\cos x+\tan x-\tan x\cos x=$$

$$=\operatorname{sen} x-\operatorname{sen} x\cos x+\frac{\operatorname{sen} x}{\cos x}-\frac{\operatorname{sen} x}{\cancel{\cos x}}\cancel{\cos x}=$$

$$=\frac{\operatorname{sen} x\cos x-\operatorname{sen} x\cos^2 x+\operatorname{sen} x-\operatorname{sen} x\cos x}{\cos x}=$$

$$=\frac{-\operatorname{sen} x\cos^2 x+\operatorname{sen} x}{\cos x}=\frac{\operatorname{sen} x\left(-\cos^2 x+1\right)}{\cos x}=\frac{\operatorname{sen} x\times\operatorname{sen}^2 x}{\cos x}=$$

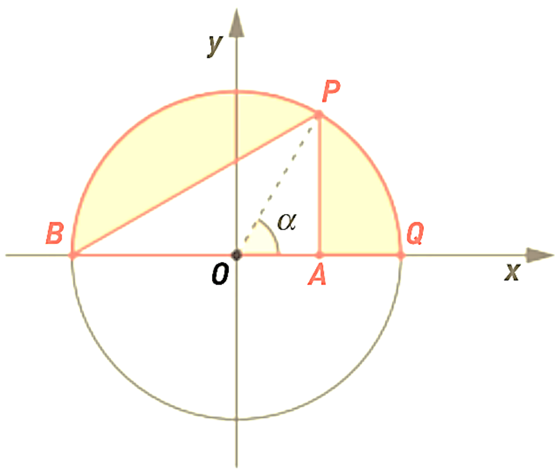
$$=\frac{\operatorname{sen}^3 x}{\cos x} \qquad \text{c.q.m.}$$

7. Na figura está representada, num referencial o.n. Oxy, uma semicircunferência de centro O e raio 1.

O ponto P move-se ao longo da semicircunferência, a partir do ponto Q, até B.

Sabe-se que:

- $Q\hat{O}P=\alpha$ radianos;
- o ponto A é a projeção ortogonal de P sobre o eixo Ox.



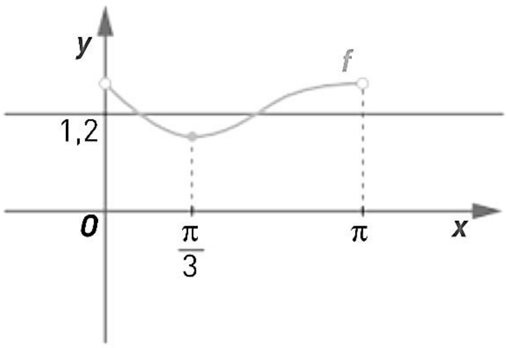
7.1. Determine a área colorida da figura para $\alpha=\dfrac{\pi}{6}$.

$$A_{\text{Colorida}}=A_{\text{Semicircunferência}}-A_{[ABP]}=\frac{\pi\times 1^2}{2}-\frac{\left(1-\cos\frac{\pi}{6}\right)\times\operatorname{sen}\frac{\pi}{6}}{2}=\frac{\pi}{2}-\frac{\left(1+\frac{\sqrt{3}}{2}\right)\times\frac{1}{2}}{2}=$$

$$=\frac{\pi}{2}-\frac{\frac{2+\sqrt{3}}{4}}{2}=\frac{\pi}{2}-\frac{2+\sqrt{3}}{8}=\frac{4\pi-2-\sqrt{3}}{8}$$

7.2. Seja f a função que a cada valor de $\alpha \in \left]0, \pi\right[$ faz corresponder a área da zona colorida da figura e que se encontra representada no referencial da figura seguinte.

7.2.1. Mostre que $f(\alpha) = \dfrac{\pi - \operatorname{sen}\alpha - \operatorname{sen}\alpha\cos\alpha}{2}$.



$$f(\alpha) = \frac{\pi \times 1^2}{2} - \frac{(1-\cos\alpha)\times\operatorname{sen}\alpha}{2} =$$

$$= \frac{\pi}{2} - \frac{\operatorname{sen}\alpha - \operatorname{sen}\alpha\cos\alpha}{2} =$$

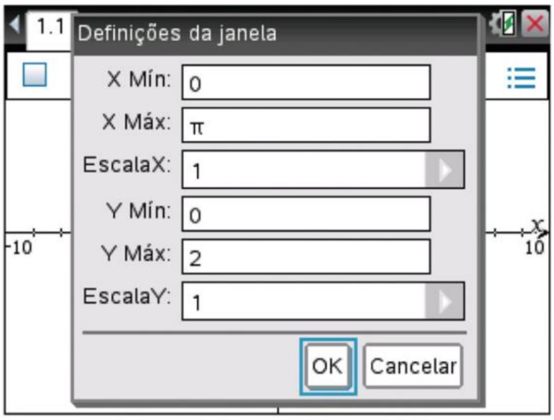
$$= \frac{\pi - \operatorname{sen}\alpha - \operatorname{sen}\alpha\cos\alpha}{2} \quad \text{c.q.m.}$$

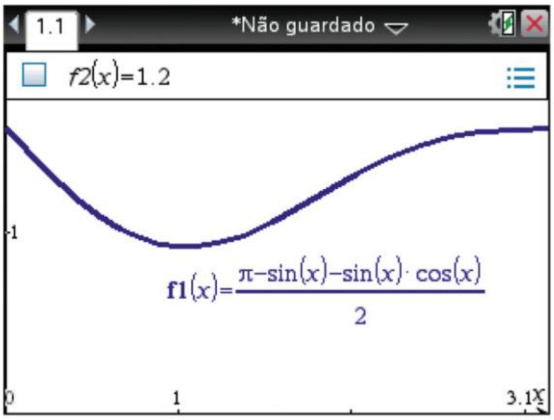
7.2.2. O ponto correspondente ao mínimo da função tem abcissa $\dfrac{\pi}{3}$.

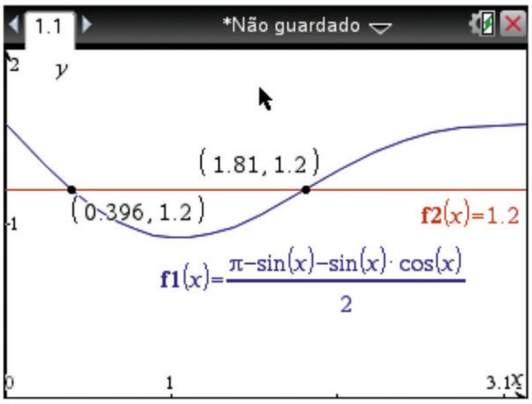
Determine a ordenada e interprete as coordenadas desse ponto no contexto apresentado.

$$f\left(\frac{\pi}{3}\right) = \frac{\pi - \operatorname{sen}\frac{\pi}{3} - \operatorname{sen}\frac{\pi}{3}\cos\frac{\pi}{3}}{2} = \frac{\pi - \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\times\frac{1}{2}}{2} = \frac{\frac{4\pi}{4} - \frac{2\sqrt{3}}{4} - \frac{\sqrt{3}}{4}}{2} = \frac{4\pi - 3\sqrt{3}}{8}$$

7.2.3. Com recurso a uma calculadora gráfica, determine os valores aproximados às décimas das amplitudes de α, em radianos, para os quais a área é de 1,2







Portanto, $\alpha \approx 0{,}4$ rad ou $\alpha \approx 1{,}8$ rad