TRIGONOMETRIA – GENERALIZAÇÃO DA FÓRMULA FUNDAMENTAL

1. Sabendo que $\textbf{sen}\,\alpha = -\frac{1}{3}$ e que $\alpha \in \left]\pi, \frac{3\pi}{2}\right[$, calcula o valor exato de:

1.1. $\cos\alpha$

$$\sin^2\alpha + \cos^2\alpha = 1 \iff \cos^2\alpha = 1 - \left(-\frac{1}{3}\right)^2 \iff \cos^2\alpha = \frac{8}{9}$$

Como $\alpha \in 3.^\circ$ Q., tem-se: $\cos\alpha = -\sqrt{\frac{8}{9}} = -\frac{2\sqrt{2}}{3}$

1.2. $\tan\alpha$

$$\tan\alpha = \frac{-\frac{1}{3}}{-\frac{2\sqrt{2}}{3}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$$

2. Determina o valor exato de $\tan\alpha - \textbf{sen}\,\alpha$, sabendo que $\cos\alpha = -\frac{1}{4}$ e que $\alpha \in \left]\frac{\pi}{2}, \pi\right[$

$$\sin^2\alpha + \cos^2\alpha = 1 \iff \sin^2\alpha = 1 - \left(-\frac{1}{4}\right)^2 \iff \sin^2\alpha = \frac{15}{16}$$

Como $\alpha \in 2.^\circ$ Q., tem-se: $\sin\alpha = \sqrt{\frac{15}{16}} = \frac{\sqrt{15}}{4}$

$$\tan\alpha = \frac{\frac{\sqrt{15}}{4}}{-\frac{1}{4}} = -\sqrt{15}$$

$$\tan\alpha - \sin\alpha = -\sqrt{15} - \frac{\sqrt{15}}{4} = -\frac{5\sqrt{15}}{4}$$

3. Mostra que sempre que as expressões têm significado, se tem:

3.1. $(\cos\beta - \text{sen}\,\beta)^2 = 2 - (\cos\beta + \sin\beta)^2$

$$(\cos\beta - \sin\beta)^2 = \cos^2\beta - 2\cos\beta\sin\beta + \sin^2\beta =$$

$$= (\cos^2\beta + \sin^2\beta) - 2\cos\beta\sin\beta = 1 - 2\cos\beta\sin\beta$$

$$= 2 - (1 + 2\cos\beta\sin\beta) = 2 - (\cos^2\beta + 2\cos\beta\sin\beta + \sin^2\beta)$$

$$= 2 - (\cos\beta + \sin\beta)^2$$

3.2. $\frac{\cos^2\alpha}{1+\text{sen}\,\alpha} = 1 - \text{sen}\,\alpha$

$$\frac{\cos^2\alpha}{1+\sin\alpha} = \frac{1-\sin^2\alpha}{1+\sin\alpha} = \frac{(1-\sin\alpha)(1+\sin\alpha)}{1+\sin\alpha} = 1 - \sin\alpha$$

3.3. $\tan\theta + \frac{1}{\tan\theta} = \frac{1}{\text{sen}\,\theta \times \cos\theta}$

$$\tan\theta + \frac{1}{\tan\theta} = \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} = \frac{\sin^2\theta + \cos^2\theta}{\sin\theta \times \cos\theta} =$$

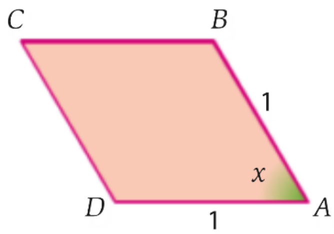
$$= \frac{1}{\sin\theta \times \cos\theta}$$

3.4. $\frac{\text{sen}\,\beta}{1+\cos\beta} = \frac{1-\cos\beta}{\text{sen}\,\beta}$

$$\frac{\sin\beta}{1+\cos\beta} = \frac{\sin\beta(1-\cos\beta)}{(1+\cos\beta)(1-\cos\beta)} = \frac{\sin\beta(1-\cos\beta)}{1-\cos^2\beta} =$$

$$= \frac{\sin\beta(1-\cos\beta)}{\sin^2\beta} = \frac{1-\cos\beta}{\sin\beta}$$

4. O losango $[ABCD]$, representado na figura, tem lado unitário. O ângulo DAB tem amplitude de x radianos ($x \in]0, \pi[$).



4.1. Mostra que a área do losango é dada, em função de x, pela seguinte expressão analítica:

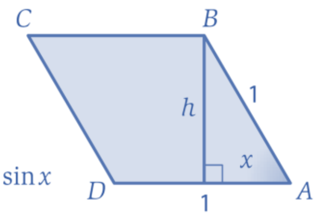
$$A(x) = \textbf{sen}\, x$$

Sugestão: Na determinação da área do losango, considera-o um paralelogramo.

$A(x) = \overline{AD} \times h$

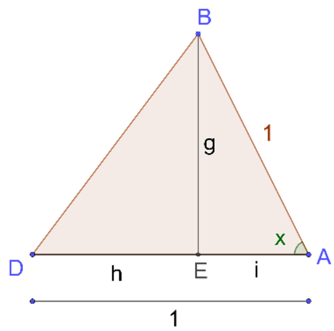
$\sin x = \frac{h}{1} \Leftrightarrow h = \sin x$

$A(x) = 1 \times \sin x \Leftrightarrow A(x) = \sin x$



4.2. Considera $\overline{DB} = 2 - \sqrt{2}$

4.2.1 Determina $\cos x$



Consideremos parte do losango $[ABCD]$

$\overline{DB} = 2 - \sqrt{2}$

$\cos x = \frac{i}{1} \Leftrightarrow \cos x = i$

$h = 1 - i \Leftrightarrow h = 1 - \cos x$

$\text{sen}\, x = \frac{g}{1} \Leftrightarrow \text{sen}\, x = g$

Então, pelo Teorema de Pitágoras

$\overline{DB}^2 = g^2 + h^2 \Leftrightarrow$

$\Leftrightarrow \left(2 - \sqrt{2}\right)^2 = \text{sen}^2\, x + (1 - \cos x)^2$

$\Leftrightarrow 4 - 4\sqrt{2} + 2 = \text{sen}^2\, x + 1 - 2\cos x + \cos^2 x$

$\Leftrightarrow 6 - 4\sqrt{2} = \text{sen}^2\, x + \cos^2 x + 1 - 2\cos x$

$\text{sen}^2\, x + \cos^2 x = 1$

$\Leftrightarrow 6 - 4\sqrt{2} = 1 + 1 - 2\cos x$

$\Leftrightarrow 4 - 4\sqrt{2} = -2\cos x$

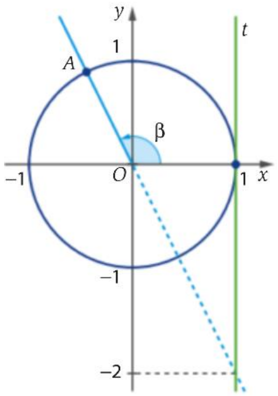
$\Leftrightarrow \cos x = 2\sqrt{2} - 2$

4.2.2 Calcula o valor exato da área do losango.

$$\sin^2 x+\cos^2 x=1 \Leftrightarrow \sin^2 x=1-\left(-2+2\sqrt{2}\right)^2 \Leftrightarrow$$

$$\Leftrightarrow \sin^2 x=-11+8\sqrt{2} \underset{x\in\left]0,\ \pi\right[}{\Leftrightarrow} \sin x=\sqrt{-11+8\sqrt{2}}$$

5. Na figura ao lado estão representados, num referencial o.n. do plano:

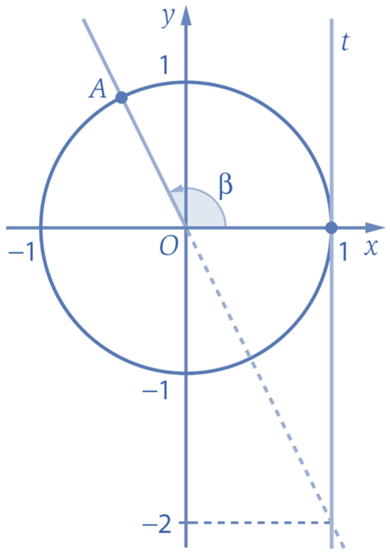


- a circunferência trigonométrica;
- um ângulo, de amplitude β, que tem por lado origem o semieixo positivo das abcissas e por outro lado extremidade a semirreta $\dot{O}A$;
- a reta t , de equação $x = 1$

Determina o valor exato da expressão

$$\cos^2\beta - \tan\beta - \text{sen}\,\beta$$

A reta OA interseta o eixo das tangentes no ponto deste eixo de ordenada -2, pelo que $\tan\beta=-2$.



$$\tan^2\beta+1=\frac{1}{\cos^2\beta} \Leftrightarrow (-2)^2+1=\frac{1}{\cos^2\beta} \Leftrightarrow \cos^2\beta=\frac{1}{5}$$

$$\sin^2\beta+\cos^2\beta=1 \Leftrightarrow \sin^2\beta=1-\frac{1}{5} \Leftrightarrow \sin^2\beta=\frac{4}{5}$$

Como $\beta\in 2.^\circ\,Q$, tem-se: $\sin\beta=\sqrt{\frac{4}{5}}=\frac{2}{\sqrt{5}}=\frac{2\sqrt{5}}{5}$

Concluindo:

$$\cos^2\beta-\tan\beta-\sin\beta=\frac{1}{5}-(-2)-\frac{2\sqrt{5}}{5}=\frac{11-2\sqrt{5}}{5}$$

6. Mostra que, no seu domínio, são universais as seguintes condições:

6.1. $(1-\text{sen}\,\alpha)(1+\text{sen}\,\alpha)=\cos^2\alpha$

$$(1-\sin\alpha)(1+\sin\alpha) = 1-\sin^2\alpha = 1-(1-\cos^2\alpha) = \cos^2\alpha$$

6.2. $\frac{1}{\cos^2\beta}-\cos^2\beta=\text{sen}^2\,\beta\left(\frac{1}{\cos^2\beta}+1\right)$

$$\sin^2\beta\left(\frac{1}{\cos^2\beta}+1\right) = \frac{\sin^2\beta}{\cos^2\beta}+\sin^2\beta = \tan^2\beta+\sin^2\beta =$$

$$= \frac{1}{\cos^2\beta}-1+\sin^2\beta = \frac{1}{\cos^2\beta}-\cos^2\beta$$

Alternativamente, podemos fazer:

$$\frac{1}{\cos^2\beta}-\cos^2\beta = \frac{\sin^2\beta+\cos^2\beta}{\cos^2\beta}-\cos^2\beta = \frac{\sin^2\beta}{\cos^2\beta}+1-\cos^2\beta = \frac{\sin^2\beta}{\cos^2\beta}+\sin^2\beta+\cos^2\beta-\cos^2\beta =$$

$$=\sin^2\beta\left(\frac{1}{\cos^2\beta}+1\right)$$