1. Considera a função f definida em $\mathbb{R}$ por $f(x) = 1 - 2\sin\left(\frac{\pi}{5} + \frac{2}{3}x\right)$.

1.1. Determina o contradomínio da função f.

Contradomínio da função sin $[-1, 1]$

$-1 \leq \sin\left(\frac{\pi}{5} + \frac{2}{3}x\right) \leq 1 \Leftrightarrow$

$\Leftrightarrow -2 \leq 2\sin(\pi/5 + 2/3\,x) \leq 2$

$\Leftrightarrow 2 \geq -2\sin(\pi/5 + 2/3\,x) \geq -2$

$\Leftrightarrow -2 \leq -2\sin(\pi/5 + 2/3\,x) \leq 2$

$\Leftrightarrow 1 - 2 \leq 1 - 2\sin(\pi/5 + 2/3\,x) \leq 2 + 1$

$\Leftrightarrow -1 \leq 1 - 2\sin(\pi/5 + 2/3\,x) \leq 3$

logo $D'_f = [-1, 3]$

1.2. Escreve a expressão geral dos zeros de f.

$f(x) = 0 \Leftrightarrow 1 - 2\sin(\pi/5 + 2/3\,x) = 0$

$\Leftrightarrow -2\sin(\pi/5 + 2/3\,x) = -1$

$\Leftrightarrow \sin(\pi/5 + 2/3\,x) = \frac{1}{2}$

$\Leftrightarrow \sin(\pi/5 + 2/3\,x) = \sin(\pi/6)$

$\Leftrightarrow \pi/5 + 2/3\,x = \pi/6 + 2k\pi \vee \pi/5 + 2/3\,x = \pi - \pi/6 + 2k\pi$, $k \in \mathbb{Z}$

$\Leftrightarrow \frac{2}{3}x = -\frac{\pi}{30} + 2k\pi \vee \frac{2}{3}x = \frac{19}{30}\pi + 2k\pi$, $k \in \mathbb{Z}$

$\Leftrightarrow x = -\frac{\pi}{20} + 3k\pi \vee x = \frac{19}{20}\pi + 3k\pi$, $k \in \mathbb{Z}$

1.3. Prova que f é periódica de período 3π .

$$f(x+3\pi)= 1-2\sin\left(\frac{\pi}{3}+\frac{2}{3}(x+3\pi)\right)=$$
$$= 1-2\sin\left(\frac{\pi}{3}+\frac{2}{3}x+2\pi\right)=$$
$$= 1-2\sin\left(\pi/3+2/3\, x\right)=f(x)$$

logo período é 3π

2. Considera as funções *f*, *g*, *h*, *i*, *j*, *k* e *l* definidas, respetivamente, por:

$f(x)=5+3\sin x$ $\quad g(x)=1-\sin 2x$ $\quad h(x)=3-6\cos\frac{1}{2}x$ $\quad i(x)=\frac{2}{3+\cos x}$

$J(x)=2\tan\left(x+\frac{\pi}{3}\right)$ $\quad k(x)=1+\tan\left(5x+\frac{\pi}{6}\right)$ $\quad l(x)=3-4\cos^2\left(2x+\frac{\pi}{4}\right)$

2.1. Determina o domínio e o contradomínio de cada uma das funções.

$\underline{\underline{f(x)}}$ $\quad D_f=\mathbb{R}$

$$-1\leq \sin x\leq 1$$
$$-3\leq 3\sin x\leq 3$$
$$5-3\leq 5+3\sin x\leq 5+3$$
$$2\leq f(x)\leq 8$$
$$D'_f=[2,8]$$

$\underline{\underline{g(x)}}$ $\quad D_g=\mathbb{R}$

$$-1\leq \sin(2x)\leq 1$$
$$1\geq -\sin(2x)\geq -1$$
$$-1\leq -\sin(2x)\leq 1 \rightarrow 1-1\leq f(x)\leq 1+1$$
$$0\leq g(x)\leq 2$$
$$D'_g=[0,2]$$

$\underline{h(x)}$ $D_h = \mathbb{R}$

$-1 \leq \cos\left(-\frac{1}{2}x\right) \leq 1$

$-6 \leq 6\cos\left(\frac{1}{2}x\right) \leq 6$

$6 \geq -6\cos\left(\frac{1}{2}x\right) \geq -6$

$-6 \leq -6\cos\left(\frac{1}{2}x\right) \leq 6$

$3-6 \leq 3-6\cos\left(\frac{1}{2}x\right) \leq 6+3$

$-3 \leq h(x) \leq 9$

$D'_h = [-3, 9]$

$\underline{i(x)}$ $3+\cos x \neq 0$

$\cos x \neq -3 \quad , \forall x \in \mathbb{R}$

logo $D_i = \mathbb{R}$

$-1 \leq \cos x \leq 1$

$2 \leq 3+\cos x \leq 4$

$\frac{1}{2} \geq \frac{1}{3+\cos x} \geq \frac{1}{4}$

$\frac{1}{4} \leq \frac{1}{3+\cos x} \leq \frac{1}{2}$

$\rightarrow \frac{1}{2} \leq \frac{2}{3+\cos x} \leq 1$

$D'_i = \left[\frac{1}{2}, 1\right]$

$g(x)$ $D_g = \{x \in \mathbb{R} : x + \frac{\pi}{3} \neq \frac{\pi}{2} + k\pi, k \in \mathbb{Z}\}$

$x + \frac{\pi}{3} = \frac{\pi}{2} + k\pi, k \in \mathbb{Z}$

$x = \frac{\pi}{2} - \frac{\pi}{3} + k\pi, k \in \mathbb{Z}$

$x = \frac{\pi}{6} + k\pi, k \in \mathbb{Z}$

logo

$D_g = \{x \in \mathbb{R} : x \neq \frac{\pi}{6} + k\pi, k \in \mathbb{Z}\}$

$D'_g = \mathbb{R}$

$k(x)$ $D_k = \{x \in \mathbb{R} : 5x + \frac{\pi}{6} \neq \frac{\pi}{2} + k\pi, k \in \mathbb{Z}\}$

$5x + \frac{\pi}{6} = \frac{\pi}{2} + k\pi, k \in \mathbb{Z}$

$\Leftrightarrow 5x = \frac{\pi}{2} - \frac{\pi}{6} + k\pi, k \in \mathbb{Z}$

$\Leftrightarrow 5x = \frac{\pi}{3} + k\pi, k \in \mathbb{Z}$

$\Leftrightarrow x = \frac{\pi}{15} + k\frac{\pi}{5}, k \in \mathbb{Z}$

logo

$D_k = \{x \in \mathbb{R} : x \neq \frac{\pi}{15} + k\frac{\pi}{5}, k \in \mathbb{Z}\}$

$D'_k = \mathbb{R}$

$l(x)$ $D_l = \mathbb{R}$

Nota
$-1 \leq \cos x \leq 1$
$0 \leq \cos^2 x \leq 1$

$0 \leq \cos^2(2x + \frac{\pi}{4}) \leq 1$

$0 \leq 4\cos^2(2x + \frac{\pi}{4}) \leq 4$

$-4 \leq -4\cos^2(2x + \pi/4) \leq 0$

$3 - 4 \leq 3 - 4\cos^2(2x + \pi/4) \leq 3$

$-1 \leq l(x) \leq 3$

$D'_l = [-1, 3]$

2.2. Determina, caso existam, uma expressão geral dos zeros de cada uma das funções.

$f(x)$ Como $D'_f = [2, 8]$, a função não tem zeros.

$g(x)$

$g(x) = 0 \Leftrightarrow 1 - \operatorname{sen}(2x) = 0 \Leftrightarrow$

$\Leftrightarrow \operatorname{sen}(2x) = 1$

$\Leftrightarrow \operatorname{sen}(2x) = \operatorname{sen}(\frac{\pi}{2})$

$\Leftrightarrow 2x = \frac{\pi}{2} + 2k\pi, \ k \in \mathbb{Z}$

$\Leftrightarrow x = \frac{\pi}{4} + k\pi, \ k \in \mathbb{Z}$

$\underline{\underline{h(x)}} = 0 \Leftrightarrow 3 - 6\cos\left(\frac{1}{2}x\right) = 0$

$\Leftrightarrow \cos\left(\frac{1}{2}x\right) = \frac{1}{2}$

$\Leftrightarrow \cos\left(\frac{1}{2}x\right) = \cos\left(\frac{\pi}{3}\right)$

$\Leftrightarrow \frac{1}{2}x = \frac{\pi}{3} + 2k\pi \vee \frac{1}{2}x = -\frac{\pi}{3} + 2k\pi, k \in \mathbb{Z}$

$\Leftrightarrow x = \frac{2\pi}{3} + 4k\pi \vee x = -\frac{2\pi}{3} + 4k\pi, k \in \mathbb{Z}$

$\underline{\underline{i(x)}}$ como $D_i' = [1/2, 1]$ então $i(x)$ não tem zeros.

$\underline{\underline{j(x)}} = 0 \Leftrightarrow 2\tan\left(x + \frac{\pi}{3}\right) = 0 \Leftrightarrow$

$\Leftrightarrow \tan\left(x + \frac{\pi}{3}\right) = 0$

$\Leftrightarrow \tan\left(x + \frac{\pi}{3}\right) = \tan 0$

$\Leftrightarrow x + \frac{\pi}{3} = k\pi, k \in \mathbb{Z}$

$\Leftrightarrow x = -\frac{\pi}{3} + k\pi, k \in \mathbb{Z}$

$\underline{\underline{k(x) = 0}} \Leftrightarrow 1 + \tan\left(5x + \frac{\pi}{6}\right) = 0$

$\Leftrightarrow \tan\left(5x + \frac{\pi}{6}\right) = -1$

$\Leftrightarrow \tan\left(5x + \frac{\pi}{6}\right) = -\tan\left(\frac{\pi}{4}\right)$

$\Leftrightarrow 5x + \frac{\pi}{6} = -\frac{\pi}{4} + k\pi, k \in \mathbb{Z}$

$\Leftrightarrow x = \frac{\pi}{60} + \frac{k\pi}{5}, k \in \mathbb{Z}$

$\underline{\underline{l(x)=0}} \Leftrightarrow 3-4\cos^2\left(2x+\frac{\pi}{4}\right)=0$

$\Leftrightarrow \cos^2\left(2x+\frac{\pi}{4}\right)=\frac{3}{4}$

$\Leftrightarrow \cos\left(2x+\frac{\pi}{4}\right)=\pm\sqrt{\frac{3}{4}}$

$\Leftrightarrow \cos\left(2x+\frac{\pi}{4}\right)=\frac{\sqrt{3}}{2} \vee \cos\left(2x+\frac{\pi}{4}\right)=-\frac{\sqrt{3}}{2}$

$\Leftrightarrow \cos\left(2x+\frac{\pi}{4}\right)=\cos\left(\frac{\pi}{6}\right) \vee \cos\left(2x+\frac{\pi}{4}\right)=\cos\left(-\frac{\pi}{6}\right)$

$\Leftrightarrow 2x+\frac{\pi}{4}=\frac{\pi}{6}+2k\pi \vee 2x+\frac{\pi}{4}=-\frac{\pi}{6}+2k\pi,\ k\in\mathbb{Z}$

$\Leftrightarrow x=-\frac{\pi}{24}+k\pi \vee x=-\frac{5\pi}{24}+k\pi,\ k\in\mathbb{Z}$

2.3. Estuda, quanto à paridade, as funções *f*, *g*, *h* e *i* .

$\underline{\underline{f(x)}}$

$D_f=\mathbb{R}$

$f(-x)=5+3\sin(-x)=5-3\sin(x)$

logo f não é par nem ímpar

$\underline{\underline{g(x)}}$

$D_g=\mathbb{R}$

$g(-x)=1-\operatorname{sen}(-2x)=1+\operatorname{sen}(2x)$

Assim $g(-x)\neq g(x)$ e

$g(-x)\neq -g(x)$

logo g não é par nem ímpar

$\underline{h(x)}$ $D_h = \mathbb{R}$

$h(-x) = 3 - 6\cos\left(-\frac{1}{2}x\right) =$

$= 3 - 6\cos\left(\frac{1}{2}x\right) = h(x)$

logo h é par

$\underline{i(x)}$ $D_i = \mathbb{R}$

$i(-x) = \frac{2}{3+\cos(-x)} = \frac{2}{3+\cos x} = i(x)$

logo i é par

2.4. Escreve uma expressão geral dos maximizantes das funções *g*, *i* e *l* .

$\underline{g(x)}$ $D'_g = [0, 2]$

$g(x) = 2 \Leftrightarrow 1 - \operatorname{sen}(2x) = 2$

$\Leftrightarrow \operatorname{sen}(2x) = -1$

$\Leftrightarrow \operatorname{sen}(2x) = \operatorname{sen}\left(-\frac{\pi}{2}\right)$

$\Leftrightarrow 2x = -\frac{\pi}{2} + 2k\pi, \ k \in \mathbb{Z}$

$\Leftrightarrow x = -\frac{\pi}{4} + k\pi, \ k \in \mathbb{Z}$

$i(x)$ $D'_i = \left[\frac{1}{2}, 1\right]$

$i(x) = 1 \Leftrightarrow \frac{2}{3+\cos x} = 1 \Leftrightarrow$

$\Leftrightarrow 3 + \cos x = 2$

$\Leftrightarrow \cos x = -1$

$\Leftrightarrow x = -\pi + 2k\pi, \ k \in \mathbb{Z}$

$l(x)$ $D'_l = [-1, 3]$

$l(x) = 3 \Leftrightarrow 3 - 4\cos^2\left(2x + \frac{\pi}{4}\right) = 3$

$\Leftrightarrow -4\cos^2\left(2x + \frac{\pi}{4}\right) = 0$

$\Leftrightarrow \cos^2\left(2x + \pi/4\right) = 0$

$\Leftrightarrow \cos\left(2x + \pi/4\right) = 0$

$\Leftrightarrow \cos\left(2x + \pi/4\right) = \cos(\pi/2)$

$\Leftrightarrow 2x + \pi/4 = \pi/2 + k\pi, \ k \in \mathbb{Z}$

$\Leftrightarrow x = \pi/8 + \frac{k\pi}{2}, \ k \in \mathbb{Z}$

2.5. Escreve uma expressão geral dos minimizantes das funções *f*, *h* e *i* .

$f(x)$ $D'_f = [2, 8]$

$f(x) = 2 \Leftrightarrow 5 + 3\,\text{sen}\, x = 2 \Leftrightarrow \text{sen}\, x = -1$

$\Leftrightarrow x = -\frac{\pi}{2} + 2k\pi, \ k \in \mathbb{Z}$

$\underline{h(x)}$ $D'_h = [-3, 9]$

$h(x) = -3 \Leftrightarrow 3 - 6\cos\left(\frac{1}{2}x\right) = -3$

$\Leftrightarrow \cos\left(\frac{1}{2}x\right) = 1$

$\Leftrightarrow \frac{1}{2}x = 2k\pi, k \in \mathbb{Z}$

$\Leftrightarrow x = 4k\pi, k \in \mathbb{Z}$

$\underline{\underline{l(x)}}$ $D'_l = [-1, 3]$

$l(x) = -1 \Leftrightarrow 3 - 4\cos^2\left(2x + \frac{\pi}{4}\right) = -1$

$\Leftrightarrow \cos^2\left(2x + \pi/4\right) = 1$

$\Leftrightarrow \cos\left(2x + \frac{\pi}{4}\right) = 1 \vee \cos\left(2x + \frac{\pi}{4}\right) = -1$

$\Leftrightarrow 2x + \frac{\pi}{4} = k\pi, \quad k \in \mathbb{Z}$

$\Leftrightarrow 2x = -\frac{\pi}{4} + k\pi, \quad k \in \mathbb{Z}$

$\Leftrightarrow x = -\frac{\pi}{8} + \frac{k\pi}{2}, \quad k \in \mathbb{Z}$

2.6. Prova que:

2.6.1. g é periódica de período π .

$$g(x+\pi)=1-\operatorname{sen}(2(x+\pi))=1-\operatorname{sen}(2x+2\pi)$$
$$=1-\operatorname{sen}(2x)=g(x)$$

g tem período π

2.6.2. h é periódica de período 4π .

$$h(x+4\pi)=3+6\cos\left(\frac{1}{2}(x+4\pi)\right)=$$
$$=3+6\cos\left(\frac{1}{2}x+2\pi\right)=$$
$$=3+6\cos\left(\frac{1}{2}x\right)=h(x)$$

h tem período 4π

2.6.3. k é periódica de período $\frac{\pi}{5}$.

$$k\left(x+\frac{\pi}{5}\right)=1+\tan\left(5\left(x+\frac{\pi}{5}\right)+\frac{\pi}{6}\right)=$$
$$=1+\tan\left(5x+\pi+\frac{\pi}{6}\right)=$$
$$=1+\tan\left(5x+\frac{\pi}{6}+\pi\right)=$$
$$=1+\tan\left(5x+\frac{\pi}{6}\right)=k(x)$$

k tem período $\frac{\pi}{5}$

3. Determina, caso existam, os valores de $x \in [-\pi , \pi[$ tais que:

3.1. $\cos^2 x = \frac{1}{4}$ **3.2.** $2\sin 2x = -1$ **3.3.** $3\tan\left(x + \frac{\pi}{3}\right) = 3$

3.4. $\sin x\,(1 - 2\cos x) = 0$ **3.5.** $2\sin x \cos x + \sin x = 0$ **3.6.** $2\sin^2 x - \sin x - 1 = 0$

4. Sem recurso à calculadora, determina o valor exato das seguintes expressões.

4.1. $\arcsin\frac{\sqrt{2}}{2}$ **4.2.** $\arctan\frac{\sqrt{3}}{3}$ **4.3.** $\arcsin\left(\sin\frac{\pi}{5}\right)$

4.4. $\cos\left(\arcsin\frac{\sqrt{3}}{2}\right)$ **4.5.** $\arccos\left(\sin\frac{2}{3}\pi\right)$ **4.6.** $\tan\left(\arcsin\left(-\frac{1}{2}\right)\right)$

Soluções

1.

1.1. $D_f{}' = [-1\,,3]$

1.2. $x = -\frac{\pi}{20} + 3k\pi \ \vee \ x = \frac{19}{20}\pi + 3k\pi\,, k \in \mathbb{Z}$

2.

2.1. $D_f = \mathbb{R} \quad D'_f = [2\,,8] \qquad D_g = \mathbb{R} \quad D'_g = [0\,,2] \qquad D_h = \mathbb{R} \quad D'_h = [-3\,,9] \qquad D_i = \mathbb{R} \quad D'_i = \left[\frac{1}{2}\,,1\right]$

$D_j = \mathbb{R}\backslash\left\{x\colon x = \frac{\pi}{6} + k\pi\,, k \in \mathbb{Z}\right\} \quad D'_j = \mathbb{R} \qquad D_k = \mathbb{R}\backslash\left\{x\colon x = \frac{\pi}{15} + k\frac{\pi}{3}\,, k \in \mathbb{Z}\right\} \quad D'_k = \mathbb{R}$

$D_l = \mathbb{R} \quad D'_l = [-1\,,3]$

2.2. f: não tem zeros $\qquad g\colon x = \frac{\pi}{4} + k\pi, k \in \mathbb{Z} \qquad h\colon x = \frac{2\pi}{3} + 4k\pi \ \vee \ x = -\frac{2\pi}{3} + 4k\pi, k \in \mathbb{Z}$

i: não tem zeros $\qquad j\colon x = -\frac{\pi}{3} + k\pi, k \in \mathbb{Z} \qquad k\colon x = \frac{\pi}{60} + k\frac{\pi}{5}, k \in \mathbb{Z}$

$l\colon x = -\frac{\pi}{24} + k\pi \ \vee \ x = -\frac{5\pi}{24} + 4k\pi, k \in \mathbb{Z}$

2.3. f: não é par nem impar $\qquad g$: não é par nem impar $\qquad h$: é par $\qquad i$: é par

2.4. $g\colon x = -\frac{\pi}{4} + k\pi, k \in \mathbb{Z} \qquad i\colon x = \pi + 2k\pi, k \in \mathbb{Z} \qquad l\colon x = \frac{\pi}{8} + k\frac{\pi}{2}, k \in \mathbb{Z}$

2.5. $f\colon x = -\frac{\pi}{2} + 2k\pi, k \in \mathbb{Z} \qquad h\colon x = 4k\pi, k \in \mathbb{Z} \qquad l\colon x = \frac{3\pi}{8} + k\frac{\pi}{2}, k \in \mathbb{Z}$

3.

3.1. $C.S. = \left\{\frac{\pi}{3}\,, -\frac{\pi}{3}\,, \frac{2}{3}\pi\,, -\frac{2}{3}\pi\right\}$ **3.2.** $C.S. = \left\{-\frac{\pi}{12}\,, \frac{7\pi}{12}\,, \frac{11}{12}\pi\,, -\frac{5}{12}\pi\right\}$

3.3. $C.S. = \left\{-\frac{\pi}{12}\,, \frac{11\pi}{12}\right\}$ **3.4.** $C.S. = \left\{-\pi\,, -\frac{\pi}{3}\,, 0\,, \frac{\pi}{3}\right\}$

3.5. $C.S. = \left\{-\pi\,, -\frac{2\pi}{3}\,, 0\,, \frac{2\pi}{3}\right\}$ **3.6.** $C.S. = \left\{-\frac{5\pi}{6}\,, -\frac{\pi}{6}\,, 0\right\}$

4.

4.1. $\frac{\pi}{4}$ **4.2.** $\frac{\pi}{6}$ **4.3.** $\frac{\pi}{5}$ **4.4.** $\frac{1}{2}$ **4.5.** $\frac{\pi}{6}$ **4.6.** $-\frac{\sqrt{3}}{3}$