1. Seja f a função de domínio $\mathbb{R}\backslash\{-1\}$ definida por $f(x) = 1 + \frac{1}{x+1}$. Recorrendo à definição de limite segundo Heine, prova que:

a) $\lim\limits_{x\to 1} f(x) = \frac{3}{2}$

Seja (x_n) uma sucessão tal que $x_n \to 1$.

$x_n \to 1$

$x_n + 1 \to 2$

$\frac{1}{x_n+1} \to \frac{1}{2}$

$1 + \frac{1}{x_n+1} \to 1 + \frac{1}{2} = \frac{3}{2}$

Logo, $f(x_n) \to \frac{3}{2}$, ou seja, $\lim\limits_{x\to 1} f(x) = \frac{3}{2}$.

b) $\lim\limits_{x\to +\infty} f(x) = 1$

Seja (x_n) uma sucessão tal que $x_n \to +\infty$.

$x_n \to +\infty$

$x_n + 1 \to +\infty$

$\frac{1}{x_n+1} \to 0$

$1 + \frac{1}{x_n+1} \to 1$

Logo, $f(x_n) \to 1$, ou seja, $\lim\limits_{x\to +\infty} f(x) = 1$.

c) $\lim\limits_{x\to -1^+} f(x) = +\infty$

Seja (x_n) uma sucessão tal que $x_n \to -1$ e $x_n > -1$, $\forall\, n \in \mathbb{N}$.

$x_n \to -1^+$

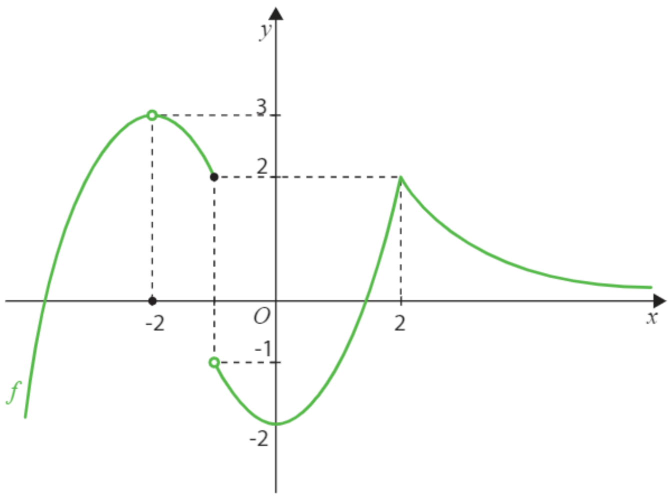
$x_n + 1 \to 0^+$

$\frac{1}{x_n+1} \to +\infty$

$1 + \frac{1}{x_n+1} \to +\infty$

Logo, $f(x_n) \to +\infty$, ou seja, $\lim\limits_{x\to -1^+} f(x) = +\infty$.

2. Na figura está representada parte do gráfico da função f . Determina:



a) $\lim f(a_n)$, sendo $a_n = \frac{2n+1}{n+2}$

$$\lim a_n = \lim \frac{2n+1}{n+2} = \lim\left(2 - \frac{3}{n+2}\right) = $$
$$= 2^-$$

Cálculo auxiliar

$2n+1$	$n+2$
$-2n-4$	2
-3	

Logo, $\lim f(a_n) = \lim\limits_{x\to 2^-} f(x) = 2$

b) $\lim f(b_n)$, sendo $b_n = \frac{-n+1}{n+2}$

$$\lim b_n = \lim \frac{-n+1}{n+2} = \lim\left(-1 + \frac{3}{n+2}\right) = -1^+$$

Cálculo auxiliar

$-n+1$	$n+2$
$n+2$	-1
3	

Logo, $\lim f(b_n) = \lim\limits_{x\to -1^+} f(x) = -1$

c) $\lim f(c_n)$, sendo $c_n = \frac{-n-2}{n+1}$

$\lim c_n = \frac{-n-2}{n+1} = \lim\left(-1 - \frac{1}{n+1}\right) = -1^-$

Cálculo auxiliar

$$\begin{array}{r|l} -n-2 & n+1 \\ \hline n+1 & -1 \\ \hline -1 & \end{array}$$

Logo, $\lim f(c_n) = \lim_{x \to -1^-} f(x) = 2$

d) $\lim f(d_n)$, sendo $d_n = \frac{-2n+1}{n+2}$

$\lim d_n = \lim \frac{-2n+1}{n+2} = \lim\left(-2 + \frac{5}{n+2}\right) = -2^+$

Cálculo auxiliar

$$\begin{array}{r|l} -2n+1 & n+2 \\ \hline +2n+4 & -2 \\ \hline 5 & \end{array}$$

Logo, $\lim f(d_n) = \lim_{x \to -2^+} f(x) = 3$

e) $\lim f(u_n)$, sendo $u_n = \frac{-n^2+1}{n+2}$

$\lim u_n = \lim \frac{-n^2+1}{n+2} = \lim \frac{n^2\left(-1+\frac{1}{n^2}\right)}{n\left(1+\frac{2}{n}\right)} = \frac{\infty(-1+0)}{1+0} = -\infty$

Logo, $\lim f(u_n) = \lim_{x \to -\infty} f(x) = -\infty$

f) $\lim f(v_n)$, sendo $v_n = \frac{n^2+1}{n+2}$

$\lim v_n = \lim \frac{n^2+1}{n+2} = \lim \frac{n^2\left(1+\frac{1}{n^2}\right)}{n\left(1+\frac{2}{n}\right)} = \frac{\infty(1+0)}{1+0} = +\infty$

Logo, , $\lim f(u_n) = \lim_{x \to +\infty} f(x) = 0$

3. Seja *f* a função representada graficamente na figura. Determina os seguintes limites.

a) $\lim\limits_{x \to -2} f(x) = 1$

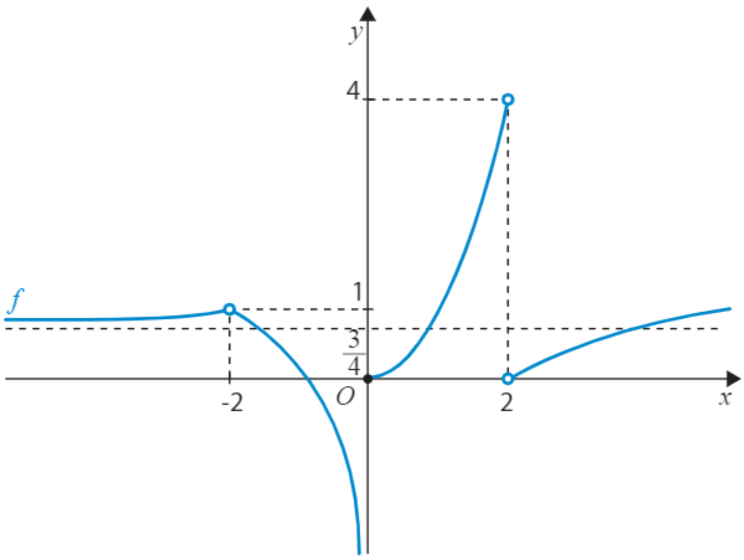
b) $\lim\limits_{x \to 0^+} f(x) = 0$

c) $\lim\limits_{x \to 0^-} f(x) = -\infty$

d) $\lim\limits_{x \to 2^+} [f(x)]^2 = 0^2 = 0$

e) $\lim\limits_{x \to 2^-} \sqrt{f(x)} = \sqrt{4} = 2$

f) $\lim\limits_{x \to -\infty} \frac{f(x)}{x} = \frac{1}{-\infty} = 0$



4. Calcula os seguintes limites, começando por identificar, caso exista, o tipo de indeterminação:

a) $\lim\limits_{x \to -\infty} (x^3 + x^2 + 2) \underset{(\infty - \infty)}{=} \lim\limits_{x \to -\infty} \left[x^3\left(1 + \frac{1}{x} + \frac{2}{x^3}\right)\right] = -\infty(1 + 0 + 0) = -\infty$

b) $\lim\limits_{x \to +\infty} \frac{x^2+2x+1}{x+1} \underset{\left(\frac{\infty}{\infty}\right)}{=} \lim\limits_{x \to +\infty} \frac{x^2\left(1+\frac{2}{x}+\frac{1}{x^2}\right)}{x\left(1+\frac{1}{x}\right)} = \frac{+\infty(1+0+0)}{1+0} = +\infty$

c) $\lim\limits_{x \to -\infty} \frac{x^2+3x}{2(x-1)(x+1)} \underset{\left(\frac{\infty}{\infty}\right)}{=} \lim\limits_{x \to -\infty} \frac{x^2\left(1+\frac{3}{x}\right)}{2x^2+2} = \lim\limits_{x \to -\infty} \frac{x^2\left(1+\frac{3}{x}\right)}{x^2\left(2+\frac{2}{x^2}\right)} = \frac{1+0}{2+0} = \frac{1}{2}$

d) $\lim\limits_{x \to +\infty} \frac{2x+1}{x^2-4x+3} \underset{\left(\frac{\infty}{\infty}\right)}{=} \lim\limits_{x \to +\infty} \frac{\frac{2}{x}+\frac{1}{x^2}}{1-\frac{4}{x}+\frac{3}{x^2}} = \frac{0+0}{1-0+0} = 0$

e) $\lim\limits_{x \to -\infty} \left(\frac{x^2-1}{x} - x\right) \underset{(\infty - \infty)}{=} \lim\limits_{x \to -\infty} \frac{x^2-1-x^2}{x} = \frac{-1}{-\infty} = 0$

f) $\lim\limits_{x\to+\infty}\frac{\frac{1}{x+2}}{\frac{2}{3x+1}} \underset{\left(\frac{0}{0}\right)}{=} \lim\limits_{x\to+\infty}\frac{3x+1}{2x+4} = \lim\limits_{x\to+\infty}\frac{3+\frac{1}{x}}{2+\frac{4}{x}} = \frac{3+0}{2+0} = \frac{3}{2}$

g) $\lim\limits_{x\to-\infty}\left(\frac{x^2+2x}{x+1}\times\frac{1}{3x}\right) \underset{(\infty\times 0)}{=} \lim\limits_{x\to-\infty}\frac{x^2+2x}{3x^2+3x} = \lim\limits_{x\to-\infty}\frac{1+\frac{2}{x}}{3+\frac{3}{x}} = \frac{1}{3}$

h) $\lim\limits_{x\to+\infty}\frac{|x-2|}{x^2-4} \underset{\left(\frac{\infty}{\infty}\right)}{=} \lim\limits_{x\to+\infty}\frac{x-2}{x^2-4} = \lim\limits_{x\to+\infty}\frac{x-2}{(x-2)(x+2)} = \frac{1}{+\infty} = 0$

i) $\lim\limits_{x\to-\infty}\frac{x^2-4}{|x-2|} \underset{\left(\frac{\infty}{\infty}\right)}{=} \lim\limits_{x\to-\infty}\frac{(x-2)(x+2)}{-x+2} = \lim\limits_{x\to-\infty}\frac{(x-2)(x+2)}{-(x-2)} = -(-\infty) = +\infty$

j) $\lim\limits_{x\to+\infty}\sqrt{x-1}-\sqrt{x+1} \underset{(\infty-\infty)}{=} \lim\limits_{x\to+\infty}\frac{(\sqrt{x-1}-\sqrt{x+1})(\sqrt{x-1}+\sqrt{x+1})}{\sqrt{x-1}+\sqrt{x+1}} = \lim\limits_{x\to+\infty}\frac{x-1-x-1}{\sqrt{x-1}+\sqrt{x+1}} = \frac{-2}{+\infty} = 0$

k) $\lim\limits_{x\to-\infty}\frac{\sqrt{x^2+1}}{2x} \underset{\left(\frac{\infty}{\infty}\right)}{=} \lim\limits_{x\to-\infty}\frac{|x|\sqrt{1+\frac{1}{x^2}}}{2x} = \lim\limits_{x\to-\infty}\frac{-x\sqrt{1+\frac{1}{x^2}}}{2x} = -\frac{1}{2}$

l) $\lim\limits_{x\to+\infty}\frac{x+3}{\sqrt{x^2+1}+\sqrt{x^2+2}} \underset{\left(\frac{\infty}{\infty}\right)}{=} \lim\limits_{x\to+\infty}\frac{x\left(1+\frac{3}{x}\right)}{|x|\left(\sqrt{1+\frac{1}{x^2}}+\sqrt{1+\frac{2}{x^2}}\right)} = \frac{1+0}{\sqrt{1}+\sqrt{1}} = \frac{1}{2}$

5. Calcula os seguintes limites, começando por identificar, caso exista, o tipo de indeterminação:

a) $\lim\limits_{x\to1}(x^2+2x+1) = 1+2+1 = 4$

b) $\lim\limits_{x\to-1}\frac{x^2+2x+1}{x+1} \underset{\left(\frac{0}{0}\right)}{=} \lim\limits_{x\to-1}\frac{(x+1)^2}{x+1} = -1+1 = 0$

c) $\lim\limits_{x\to0^+}\frac{x^2+2x+1}{x} = \frac{0+0+1}{0^+} = +\infty$

d) $\lim\limits_{x\to0}\frac{x^3+2x^2+x}{x} \underset{\left(\frac{0}{0}\right)}{=} \lim\limits_{x\to0}\frac{x(x^2+2+1)}{x} = 1$

e) $\lim\limits_{x\to 1}\frac{x^3+2x^2-x-2}{x^2-1} \underset{\left(\frac{0}{0}\right)}{=} \lim\limits_{x\to 1}\frac{(x-1)(x^2+3x+2)}{(x-1)(x+1)} = \frac{1+3+2}{1+1} = 3$

Cálculo auxiliar

	1	2	–1	–2
1		1	3	2
	1	3	2	0

g) $\lim\limits_{x\to 1}\frac{x^3-1}{x^4-1} \underset{\left(\frac{0}{0}\right)}{=} \lim\limits_{x\to 1}\frac{(x-1)(x^2+1+1)}{(x^2-1)(x^2+1)} = \lim\limits_{x\to 1}\frac{(x-1)(x^2+1+1)}{(x-1)(x+1)(x^2+1)} = \frac{1+1+1}{(1+1)(1+1)} = \frac{3}{4}$

Cálculo auxiliar

	1	0	0	–1
1		1	1	1
	1	1	1	0

g) $\lim\limits_{x\to 2}\frac{x^2-4}{\sqrt{x-2}} \underset{\left(\frac{0}{0}\right)}{=} \lim\limits_{x\to 2}\frac{(x-2)(x+2)\sqrt{x-2}}{\sqrt{x-2}\sqrt{x-2}} = \lim\limits_{x\to 2}\frac{(x-2)(x+2)\sqrt{x-2}}{(x-2)} = (2+2)\sqrt{2-2} = 0$

h) $\lim\limits_{x\to 0^+}\frac{\sqrt{x}}{x} \underset{\left(\frac{0}{0}\right)}{=} \lim\limits_{x\to 0^+}\frac{|x|\sqrt{\frac{1}{x}}}{x} = \sqrt{\frac{1}{0^+}} = +\infty$

i) $\lim\limits_{x\to 3}\frac{\sqrt{x-3}}{\sqrt{x^2-5x+6}} \underset{\left(\frac{0}{0}\right)}{=} \lim\limits_{x\to 3}\frac{\sqrt{x-3}}{\sqrt{(x-2)(x-3)}} = \lim\limits_{x\to 3}\frac{\sqrt{x-3}}{\sqrt{x-2}\sqrt{x-3}} = \frac{1}{\sqrt{3-2}} = 1$

Cálculo auxiliar

$x^2-5x+6=0 \Leftrightarrow x=\frac{5\pm\sqrt{25-24}}{2}$

$\Leftrightarrow x=2 \vee x=3$

j) $\lim\limits_{x\to 0^+}\frac{\sqrt{x}-x}{x^3-x^2} \underset{\left(\frac{0}{0}\right)}{=} \lim\limits_{x\to 0^+}\frac{(\sqrt{x}-x)(\sqrt{x}+x)}{x(x^2-x)(\sqrt{x}+x)} = \lim\limits_{x\to 0^+}\frac{x-x^2}{x(x^2-x)(\sqrt{x}+x)} = \lim\limits_{x\to 0^+}\frac{-(x^2-x)}{x(x^2-x)(\sqrt{x}+x)} = \frac{-1}{0^+} = -\infty$

k) $\lim\limits_{x\to 0}\frac{x^2+x}{\sqrt{x+1}-\sqrt{x^2+1}} \underset{\left(\frac{0}{0}\right)}{=} \lim\limits_{x\to 0}\frac{(x^2+x)(\sqrt{x+1}+\sqrt{x^2+1})}{x+1-x^2-1} = \lim\limits_{x\to 0}\frac{x(x+1)(\sqrt{x+1}+\sqrt{x^2+1})}{x-x^2} = \lim\limits_{x\to 0}\frac{x(x+1)(\sqrt{x+1}+\sqrt{x^2+1})}{x(1-x)} =$

$= \frac{(0+1)(\sqrt{0+1}+\sqrt{0+1})}{1-0} = 2$

l) $\lim\limits_{x \to 1^+} \frac{x-1}{\sqrt{x-1}-\sqrt{x^2-1}} \underset{\left(\frac{0}{0}\right)}{=} \lim\limits_{x \to 1^+} \frac{(x-1)\left(\sqrt{x-1}+\sqrt{x^2-1}\right)}{x-1-x^2+1} = \lim\limits_{x \to 1^+} \frac{(x-1)\left(\sqrt{x-1}+\sqrt{x^2-1}\right)}{x-x^2} = \lim\limits_{x \to 1^+} \frac{(x-1)\left(\sqrt{x-1}+\sqrt{x^2-1}\right)}{x(1-x)} =$

$$= \lim_{x \to 1^+} \frac{(x-1)\left(\sqrt{x-1}+\sqrt{x^2-1}\right)}{-x(-1+x)} = \frac{\sqrt{1^+-1}+\sqrt{1^+-1}}{-1^+} = \frac{0^++0^+}{-1} = 0$$

6. Para cada valor de k, a expressão seguinte define uma função de domínio $\mathbb{R}$.

$$\begin{cases} \frac{x-1}{x-2}+k & \text{se } x < 1 \\ 3 & \text{se } x = 1 \\ \frac{2x+1}{x} & \text{se } x > 1 \end{cases}$$

Determina o valor de k, para o qual existe $\lim\limits_{x \to 1} f(x)$

Como $1 \in D_f$, então $\lim\limits_{x \to 1} f(x)$ existe se e só se $\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^-} f(x) = f(1)$

$\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} \frac{2x+1}{x} = \frac{2+1}{1} = 3$

$\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^+} \left(\frac{x-1}{x-2}+k\right) = \frac{0}{-1}+k = k$

$f(1) = 3$

Logo, $\lim\limits_{x \to 1} f(x)$ existe se $k = 3$